THE AUTOMORPHISM TOWER OF GROUPS ACTING ON ROOTED TREES

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ABSTRACT. The group of isometries $\mathsf{Aut}(\mathcal{T}_n)$ of a rooted n-ary tree, and many of its subgroups with branching structure, have groups of automorphisms induced by conjugation in $\mathsf{Aut}(\mathcal{T}_n)$. This fact has stimulated the computation of the group of automorphisms of such well-known examples as the group $\mathfrak{G}$ studied by R. Grigorchuk, and the group $\ddot{\Gamma}$ studied by N. Gupta and the second author.

In this paper, we pursue the larger theme of towers of automorphisms of groups of tree isometries such as $\mathfrak{G}$ and $\ddot{\Gamma}$. We describe this tower for all subgroups of $\mathsf{Aut}(\mathcal{T}_2)$ which decompose as infinitely iterated wreath products. Furthermore, we describe fully the towers of $\mathfrak{G}$ and $\ddot{\Gamma}$.

More precisely, the tower of $\mathfrak{G}$ is infinite countable, and the terms of the tower are 2-groups. Quotients of successive terms are infinite elementary abelian 2-groups.

In contrast, the tower of $\ddot{\Gamma}$ has length 2, and its terms are $\{2,3\}$-groups. We show that $\mathsf{Aut}^2(\ddot{\Gamma})/\mathsf{Aut}(\ddot{\Gamma})$ is an elementary abelian 3-group of countably infinite rank, while $\mathsf{Aut}^3(\ddot{\Gamma}) = \mathsf{Aut}^2(\ddot{\Gamma})$.

1. INTRODUCTION

The completeness of a centerless[1] group is measured by the extent to which all its automorphisms are all inner, i.e. to which $G = \mathsf{Aut}\,G$. If G has outer automorphisms, then adjoining them to G is part of the process of completing G; indeed G embeds in $\mathsf{Aut}\,G$ as the subgroup of inner automorphisms, and $\mathsf{Aut}\,G$ is the one-step completion of G. This process can be repeated on $\mathsf{Aut}\,G$ to produce a subnormal series of automorphism groups $\mathsf{Aut}^i\,G$ for all $i \in \mathbb{N}$. Formally, the *automorphism tower* of G it is defined as follows:

Definition 1.1. Given a centerless group G, we define $\mathsf{Aut}^\alpha(G)$ for any ordinal α as follows:

$$\mathsf{Aut}^0(G) = G;$$
$$\mathsf{Aut}^{\alpha+1}(G) = \mathsf{Aut}(\mathsf{Aut}^\alpha(G));$$

and if α is a limit ordinal, then

$$\mathsf{Aut}^\alpha(G) = \bigcup_{\beta < \alpha} \mathsf{Aut}^\beta(G).$$

All these groups are also centerless, and hence form an ascending tower of groups.

We define the *height* $\tau(G)$ as the least ordinal α such that $\mathsf{Aut}^\alpha(G) = \mathsf{Aut}^{\alpha+1}(G)$; this minimal α exists by Theorem 2.1.

The group G is *complete* if $\tau(G) = 0$, i.e., if $\mathsf{Out}(G) = 1$.

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[1]in the sense that G has a trivial center

Many important groups have height one: for example, non-abelian free groups F satisfy $\mathsf{Aut}^2 F = \mathsf{Aut}\, F$; this was shown by J. Dyer and E. Formanek [9] if F has finite rank, and by V. Tolstykh [37] for general F. Mapping class groups, simple groups, and arithmetic subgroups of simple Lie groups, are also of height one [6]. The isometry group of a regular, *non-rooted* tree is complete [39].

In this paper, we are more specifically interested in the automorphism tower of groups acting on rooted, regular trees. These groups are quite interesting in that their automorphism tower turns out often to be quite short (1, 2 or ω, much less than the upper bound the successor of 2^ω given by Theorem 2.1), and can in concrete cases be described explicitly as groups acting on the original tree; that is, automorphisms of these groups are induced by conjugation by tree isometries. Therefore, we deal with the normalizer tower of a group defined as follows.

Definition 1.2. Given a subgroup inclusion $G \leq N$, we define the *normalizer tower* of G in N as

$$\mathsf{Norm}_N^0(G) = G;$$

$$\mathsf{Norm}_N^{\alpha+1}(G) = \mathsf{Norm}_N(\mathsf{Norm}_N^\alpha(G));$$

and if α is a limit ordinal, then

$$\mathsf{Norm}_N^\alpha(G) = \bigcup_{\beta < \alpha} \mathsf{Norm}_N^\beta(G).$$

To fix notation, let $\mathcal{T}$ be a d-regular rooted tree, for some integer $d \geq 2$. Let also G be a group acting on $\mathcal{T}$. Consider the d subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_d$ rooted at all vertices neighbouring the root of $\mathcal{T}$. The subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_d$ are permuted by G, and the (set-wise) stabilizer of $\mathcal{T}_i$ acts on $\mathcal{T}_i$ by restriction. Each of the subtrees $\mathcal{T}_i$ is isomorphic to $\mathcal{T}$, and therefore carries an action of G as well as a restricted action of $\mathsf{Stab}_{\mathcal{T}_i}(G)$ to the subtree $\mathcal{T}_i$. We call G *layered* if the direct product of d copies of G, each acting individually on $\mathcal{T}_1, \ldots, \mathcal{T}_d$, embeds in G, and G permutes transitively the subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. A first goal of this paper is to give a quite explicit way of calculating the automorphism tower of a layered group (see Section 4):

Theorem 1.3. *Let L be a layered group of isometries of the binary tree, set $N = \mathsf{Norm}_{\mathsf{Aut}\,\mathcal{T}}(L)$, and let N_* be the* layered closure *of N, i.e. the smallest layered group containing N. Then $\mathsf{Aut}^i(L) \leq N_*$ for all $i \leq \omega$.*

More precisely, let $\mathcal{C}$ be the smallest lattice[2] of subgroups of $\mathsf{Aut}\,\mathcal{T}$ containing L, N and closed under the operations $G \mapsto G \times G$, $G \mapsto \{(g,g) \,|\, g \in G\}$, $G \mapsto \mathsf{Norm}_N(G)$, and $(G, H) \mapsto \{g \in H \,\big|\, g^2 \in G \text{ and } [H, g] \leq G\}$ for every $G, H \in \mathcal{C}$ with $G \lhd H$. Then $\mathsf{Aut}^i(L) \in \mathcal{C}$ for all $i \in \mathbb{N}$.

Consider now a slightly weaker condition than being layered: that the intersection of the groups G and $G \times \cdots \times G$ have finite index in both groups. There are some finitely-generated examples of such groups; the best-studied are the "Grigorchuk group" $\mathfrak{G}$ acting on the binary tree [13] and the group $\ddot{\Gamma}$ acting on the ternary tree [16]. Both of these groups enjoy many extra properties, such as being torsion, having intermediate word-growth, and being just-infinite. The automorphism group of $\ddot{\Gamma}$ was described in [32].

A second goal of this paper is to describe quite explicitly all terms in the automorphism tower of these two groups and groups similar to them. The automorphism tower lies inside the full isometry group of the original tree, and the last term of the tower has a natural description:

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[2]i.e., family of groups closed under the operations $(H, K) \mapsto H \cap K$ and $(H, K) \mapsto \langle H, K \rangle$

Theorem 1.4. *The group $\ddot{\Gamma}$ defined in [16] has height 2.*

Theorem 1.5. *The Grigorchuk group $\mathfrak{G}$ has height ω, and $\mathsf{Aut}^\omega(\mathfrak{G})$ is the smallest layered subgroup of $\mathsf{Aut}\,\mathcal{T}$ containing G; equivalently, it is the subgroup of $\mathsf{Aut}\,\mathcal{T}$ generated by a copy of G acting below each vertex.*

Finally, in a more general direction, we are interested in knowing when the automorphism group of a countable group is again countable; and how far the automorphism tower of such a group grows. An example is the following result, whose proof closely follows [36, Theorem 3.3.1]. We make freely use of notions defined in that reference, in particular stationary sets (Definition 3.3.4) and Fodor's lemma (Theorem 3.3.11). Roughly speaking, a stationary set should be thought of as having positive measure, and Fodor's lemma states that a strictly decreasing function on a stationary set has a stationary fiber.

Proposition 1.6. *Let H be a countable group. Suppose $G \leq H \leq K$ with G finitely generated and $\mathsf{Cent}_K(G) = 1$. Then the terms of the normalizer tower of H in K are countable, and therefore the tower has countable height.*

Proof. Assume $G = \langle s_1, \ldots, s_k \rangle$, and denote the normalizer tower of H by $\{N_\alpha\}$; i.e. $N_0 = H$ and $N_{\alpha+1} = \mathsf{Norm}_K(N_\alpha)$ and $N_\alpha = \bigcup_{\gamma < \alpha} N_\gamma$ for limit ordinals α.

First, we show that N_α is countable for all $\alpha < \omega_1$, the first uncountable ordinal. This is clearly true for $\alpha = 0$. Then, for any $j \in \{1, \ldots, k\}$, we have $s_j^{N_{\alpha+1}} \subseteq N_\alpha$ and therefore $|s_j^{N_{\alpha+1}}| \leq \aleph_0$. This implies $[N_{\alpha+1} : \mathsf{Cent}_{N_{\alpha+1}}(s_j)] \leq \aleph_0$, and hence

$$[N_{\alpha+1} : \bigcap_{i=1}^{k} \mathsf{Cent}_{N_{\alpha+1}}(s_j)] = [N_{\alpha+1} : \mathsf{Cent}_{N_{\alpha+1}}(G)] = |N_{\alpha+1}| \leq \aleph_0.$$

Finally, if α is a countable limit ordinal, then N_α is a countable union of countable groups and is therefore countable.

Assume now by contradiction that $N_\alpha \neq N_\beta$ for all distinct $\alpha, \beta < \omega_1$. Let T be the set of countable limit ordinals, which is a stationary subset of ω_1. For each $\alpha \in T$ choose an $h_\alpha \in N_{\alpha+1} \setminus N_\alpha$. Since α is a limit ordinal and $N_\alpha \triangleleft N_{\alpha+1}$, we have

$$G^{h_\alpha} \leq N_\alpha = \bigcup_{\gamma < \alpha} N_\gamma;$$

now for each $j \in \{1, \ldots, k\}$ choose $\gamma_j < \alpha$ such that $s_j^{h_\alpha} \in N_{\gamma_j}$, and set $f(\alpha) = \max\{\gamma_j\}$. We have $f(\alpha) < \alpha$ for all $\alpha \in T$, i.e. f is a regressive function $T \to \omega_1$. By Fodor's lemma there exists $\gamma < \omega_1$ with

$$S = \{\alpha \in T \mid f(\alpha) = \gamma\} \text{ stationary.}$$

For each $\alpha \in S$ define $\phi_\alpha : G \to H_\gamma$ by $x \mapsto x^{h_\alpha}$. Any ϕ_α is determined by the images of G's generators, so

$$|\{\phi_\alpha \mid \alpha \in S\}| \leq |H_\gamma|^k < \omega_1;$$

however $|S| = \omega_1$ because S is stationary, so there are $\alpha_1 \neq \alpha_2 \in S$ with $\phi_{\alpha_1} = \phi_{\alpha_2}$; in other words, $1 \neq h_{\alpha_1} h_{\alpha_2}^{-1} \in \mathsf{Cent}_K(W) = 1$ and we have reached a contradiction. $\square$

Recall that a tree isometry $\alpha \in \mathsf{Aut}\,\mathcal{T}$ is *finite state* if its action is induced by a finite transducer, and that the set of finite-state isometries forms a group R — see page 7 for the definition.

Corollary 1.7. *Let R be the group of finite state isometries. Then $\mathsf{Aut}^i(R)$ is countable for all $i \in \mathbb{N}$, so $\tau(R)$ is countable.*

Proof. Apply Proposition 1.6 with $K = \mathsf{Aut}\,\mathcal{T}$ and G either the Grigorchuk group (if $d = 2$) or the Gupta-Sidki group (if $d \geq 3$). By Theorem 3.7, we have $\mathsf{Aut}^{\alpha}(R) = \mathsf{Norm}_K^{\alpha}(R)$. $\square$

As another illustration, consider $G \leq \mathsf{Aut}\,\mathcal{T}$ a finitely generated group acting on the binary tree $\mathcal{T} = \{\mathbb{1}, \mathbb{2}\}^*$; and assume that G contains for all $v \in \mathcal{T}$ an element g_v fixing $v\mathbb{1}$ and $v\mathbb{2}$, and having non-conjugate actions on the subtrees rooted at these vertices. Then $\mathsf{Cent}_{\mathsf{Aut}\,\mathcal{T}}(G) = 1$, and Proposition 1.6 applies.

2. The automorphism tower

For a good survey of known results on automorphism groups, see [29]. Even though the rest of the paper does not rely on the considerations in this section, we include here some relevant facts.

First, we note that the automorphism tower has been computed for some classes of groups, and indeed that in many cases the tower has small height. W. Burnside showed in [8], page 95 that $\mathsf{Aut}(G)$ is complete if and only if G is characteristic in $\mathsf{Aut}(G)$, i.e. if and only if $G \lhd \mathsf{Aut}^2(G)$. He then showed (op. cit., next page) that if G is a non-cyclic simple group, then $\tau(G) \leq 1$. Furthermore, the symmetric groups $\mathsf{Sym}(n)$ have height 0, when $n \neq 6$.

H. Wielandt showed in [38] that the tower of a centerless, finite group is finite. Little was then known on the height of infinite groups.

A. Rae and J. Roseblade showed in [28] that if G is a Černikov group, i.e. G is a finite extension of an abelian group and satisfies the minimal condition on subgroups, then $\tau(G)$ is finite. Then J. Dyer and E. Formanek showed in [9] that non-abelian free groups have height 1; i.e. $\mathsf{Out}(\mathsf{Aut}(F_n)) = 1$ for $n > 1$.

However, groups with infinite height abound among infinite groups; for instance, the infinite dihedral group $D_{\infty} = \langle a, b | a^2, b^2 \rangle$ has an outer automorphism exchanging a and b; and $\mathsf{Aut}\, D_{\infty} \cong D_{\infty}$, so $\tau(D_{\infty}) \geq \omega$. J. Hulse showed in [19] that centerless polycyclic groups have countable height (though not necessarily ω). He showed that D_{∞} has height $\omega + 1$, by computing its automorphism tower. It can be described using subgroups of $GL(2, \mathbb{Z}[\frac{1}{2}])$ as follows:

$$\mathsf{Aut}^{\alpha}(D_{\infty}) = \begin{cases} \begin{pmatrix} 1 & \frac{1}{2^{\alpha}}\mathbb{Z} \\ 0 & \pm 1 \end{pmatrix} & \text{for } \alpha < \omega, \\ \begin{pmatrix} 1 & \mathbb{Z}[\frac{1}{2}] \\ 0 & \pm 1 \end{pmatrix} & \text{for } \alpha = \omega, \\ \begin{pmatrix} 1 & \mathbb{Z}[\frac{1}{2}] \\ 0 & \pm 2^{\mathbb{Z}} \end{pmatrix} & \text{for } \alpha > \omega. \end{cases}$$

The well-definedness of the height of a centerless group is a result of S. Thomas:

Theorem 2.1 ([35]). *Let G be a centerless group. Then there exists an ordinal α, less than the successor of $2^{|G|}$, such that $\mathsf{Aut}^{\alpha}(G) = \mathsf{Aut}^{\alpha+1}(G)$.*

Furthermore, for any ordinal α there exists a centerless group G with $\tau(G) = \alpha$.

(This last statement was later improved by Just, Shelah and Thomas [20].)

If G has a non-trivial center, more care is needed, since the $\mathsf{Aut}^{\alpha}(G)$ no longer form a nested tower. In the definition of $\mathsf{Aut}^{\alpha}(G)$, one replaces the union by a directed limit. J. Hamkins proved in [17] that all groups G have a well-defined height, by actually showing that $\mathsf{Aut}^{\alpha}(G)$ is centerless for some ordinal α. Note that for groups possibly with non-trivial center there is no explicit bound on the height of the tower as a function of $|G|$, as in Theorem 2.1.

The considerations we make for the tower of automorphisms apply also to Lie algebras, if "automorphism group" is replaced by "algebra of derivations".

We cannot resist the temptation of repeating the main steps of the proof of Theorem 2.1, since they show a strong relation to groups acting on rooted trees:

Proof of Theorem 2.1 (Sketch). Define inductively the groups W_α, for ordinals α, by $W_0 = \mathbb{Z}/2$; $W_{\alpha+1} = W_\alpha \wr (\mathbb{Z}/2)$, in which W_α embeds as $W_\alpha \times 1$; and for limit ordinals α, $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$.

If α is of the form $1 + \beta$, then the normalizer tower of W_0 in W_α stabilizes after exactly β steps.

By a result of Fried and Kollár [11], there exists a (usually non-Galois) field $\mathbb{K}$ whose automorphism group is W_β. Consider then the group $G = PGL_2(\mathbb{K}) \rtimes W_0$. The automorphism tower of G parallels the normalizer tower of W_0 in W_β, in that $\mathsf{Aut}^\gamma(G) = PGL_2(\mathbb{K}) \rtimes \mathsf{Norm}^\gamma_{\mathsf{Aut}\,\mathbb{K}}(W_0)$ for all γ. It follows that the automorphism tower of G stabilizes after α steps. $\qquad\square$

There are still many open questions in the topic of group automorphisms, and in particular, as to how large the automorphism group can be.

Let us mention that there are finitely presented groups with infinitely generated automorphism group, see for instance [23], where the group $(\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]) \rtimes \mathbb{Z}^2$ is shown to have that property for an appropriate action of $\mathbb{Z}^2$; see also [25]. The groups $\mathfrak{G}, \ddot{\Gamma}$ that we consider in this paper also have the property of being finitely generated torsion groups but having an infinitely generated outer automorphism group; however, they are not finitely presented [24, 33].

2.1. **Structure of the paper.** In Section 3 we recall standard notation for groups acting on rooted trees. We also recall results by Y. Lavreniuk and V. Nekrashevych showing that, under certain conditions often satisfied in practice, the terms of the automorphism tower act on the same tree as the original group.

In Section 4, we study in some detail the automorphism tower of "layered" groups of isometries of the binary tree, and compute these towers for certain concrete examples.

In Section 5, we study the automorphism tower of the group $\ddot{\Gamma}$ determined in [16], and of its kin the groups $\dot{\Gamma}, \Gamma$ studied in [2]. The last group Γ was first studied in [10]. The main result is that, for the groups Γ and $\ddot{\Gamma}$, the tower terminates after two steps, i.e. they have height 2.

Finally, in Section 6, we study the automorphism tower of the Grigorchuk group $\mathfrak{G}$ acting on $\mathcal{T} = \{\mathbb{1}, \mathbb{2}\}^*$, and describe the terms of the tower by their action on same tree $\mathcal{T}$. The last term of the tower is shown to be the smallest layered group containing $\mathfrak{G}$.

2.2. **Notation.** We say that H is *characteristic* in G, written $H \lessdot G$, if $H^\phi = H$ for all $\phi \in \mathsf{Aut}(G)$. If G is centerless, we identify G with $\mathsf{Inn}(G)$, and then $H \lessdot G$ if and only if $H \lhd \mathsf{Aut}(G)$. We then say that H is *absolutely characteristic* in G if $H \lhd \mathsf{Aut}^\alpha(G)$ for all ordinals α.

We write $N : Q$ for a split extension of N by Q, and $N \cdot Q$ for an nonsplit extension.

As usual, we write $[a, b]$ for the commutator $a^{-1}b^{-1}ab$, and $a^b = b^{-1}ab$ for the conjugation action.

3. Groups acting on trees

Let $X = \{\mathbb{1}, \dots, \mathbb{d}\}$ be an alphabet with $d \geq 2$ elements. The *rooted tree* $\mathcal{T}$ on X has as set of vertices the free monoid X^*, namely the set of finite words $x_1 \dots x_n$, with $x_i \in X$ for all i. The tree structure is obtained by connecting $x_1 \dots x_n$ to $x_1 \dots x_n x_{n+1}$ for all choices

of $x_i \in X$; there is a distinguished *root vertex* $\emptyset$, the empty word. The *level* of the vertex $v = x_1 \ldots x_n$ is $|v| = n$. All vertices of level n form the nth *layer* X^n of $\mathcal{T}$.

The boundary $\partial \mathcal{T}$ of $\mathcal{T}$ is naturally the set X^ω of infinite sequences over X. Given a *ray* $v = x_1 x_2 \cdots \in \partial \mathcal{T}$, we denote by v_n the truncation $x_1 \ldots x_n$.

For $v \in \mathcal{T}$ of length n, we denote by $v\mathcal{T}$ the subtree of $\mathcal{T}$ spanned by all vertices vw with $w \in \mathcal{T}$. Abstractly, it is a tree rooted at v, isomorphic to $\mathcal{T}$.

Let $\mathsf{Sym}(X)$ be the symmetric group on X. It acts naturally on $\mathcal{T}$ by

$$(1) \qquad\qquad (x_1 x_2 \ldots x_n)^\sigma = x_1^\sigma x_2 \ldots x_n,$$

and we will always identify $\mathsf{Sym}(X)$ with its image in $\mathsf{Aut}\,\mathcal{T}$.

Let W denote the isomorphism group of $\mathcal{T}$. To avoid confusion with the automorphism group of W itself or of its subgroups, elements of W will be called *isometries*. Given $g \in G$, there is $\sigma_g \in \mathsf{Sym}(X)$ such that $g\sigma_g^{-1}$ fixes the vertices of the first level of $\mathcal{T}$; then restriction to subtrees $x\mathcal{T}$ for all $x \in X$ gives *states* $g@x \in W$, for all $x \in X$. Conversely, states $g@x$ and $\sigma_g \in \mathsf{Sym}(X)$ can be assembled to give $g \in W$. We therefore have a wreath product structure

$$W = W \wr_X \mathsf{Sym}(X), \quad g \mapsto (g@\mathbb{1}, \ldots, g@\mathbb{d})\sigma_g.$$

This process can be iterated; for $n \in \mathbb{N}$, $v \in X^n$ and $g \in W$ we write $g@v$ the state of g at vertex v, and ${}^n\sigma_g$ the permutation action of g on X^n. In other words, $g@v$ denotes the action g does on the subtree $v\mathcal{T}$ *before* vertex v is moved by ${}^n\sigma_g$.

There is a dual construction: for $v \in \mathcal{T}$ and $g \in W$ we write $v * g$ the isometry of $\mathcal{T}$ that acts as g on $v\mathcal{T}$ and fixes all other vertices. We then have the simple

Lemma 3.1. $@$ *is a right action of X^* on W, and $*$ is a left action: for all $g, h \in W$ and $v, w \in X^*$ we have*

$$(g@v)@w = g@(vw), \qquad\qquad v * (w * g) = (vw) * g,$$
$$(gh)@v = (g@v)(h@v^g), \qquad\qquad v * (gh) = (v * g)(v * h),$$

$$g = (v * g)@v, \qquad\qquad g = \left(\prod_{v \in X^n} v * (g@v) \right) {}^n\sigma_g,$$

where the $v * (g@v)$ mutually commute when v ranges over the nth layer X^n.

We also define a variant of the $*$-action, by defining isometries Zg for $Z \in \{\dashv, \nabla, \triangle\}^*$:

Definition 3.2. For $g \in \mathsf{Aut}\,\mathcal{T}$, we define

$$\triangle g = (g, \ldots, g), \quad \nabla g = (1, \ldots, 1, g, g^{-1}), \quad \dashv g = (1, \ldots, 1, g) = \mathbb{d} * g,$$

where a tuple $(g_1, \ldots, g_n)$ designates an element $h \in \mathsf{Aut}\,\mathcal{T}$ fixing the first level of $\mathcal{T}$ and satisfying $h@\mathbb{i} = g_i$.

For $H \leq \mathsf{Aut}\,\mathcal{T}$ and $Z \in \{\triangle, \nabla, \dashv\}$, we define ZH as $\{Zh \,|\, h \in H\}$. We also let $\times H$ denote the subgroup $H \times \cdots \times H$ generated by $\{\mathbb{i} * h \,|\, i \in X, h \in H\}$.

We will be concerned with various classes of groups which emerged in recent years as important subgroups of W. The *vertex stabilizer* $\mathsf{Stab}_G(v)$ is the subgroup of G fixing $v \in \mathcal{T}$. We make the following definitions: a subgroup G of W is

  level-transitive, if G acts transitively on X^n for all $n \in \mathbb{N}$;
  weakly recurrent, if it is level-transitive, and $\mathsf{Stab}_G(x)@x < G$ for all $x \in X$;
  recurrent, if it is level-transitive, and $\mathsf{Stab}_G(x)@x = G$ for all $x \in X$;
  saturated, if G contains for all $n \in \mathbb{N}$ a characteristic subgroup H_n fixing X^n such that
      $H_n@v$ is level-transitive for all $v \in X^n$;
  weakly branch, if G is level-transitive, and $(v * G) \cap G$ is non-trivial for all $v \in \mathcal{T}$;

weakly regular branch, if G is level-transitive, and has a non-trivial normal subgroup K with $x * K < K$ for all $x \in X$;

branch, if G is level-transitive, and $\langle (v * G) \cap G : v \in X^n \rangle$ has finite index in G for all $n \in \mathbb{N}$;

regular branch, if G is level-transitive, and has a finite-index normal subgroup K with $x * K < K$ for all $x \in X$;

pre-layered, if $x * G < G$ for all $x \in X$;

layered, if G is level-transitive and pre-layered.

The subgroup $\mathsf{Rist}_G(v) = v * G \cap G$ is called the *rigid stabilizer* of $v \in \mathcal{T}$. The *layered closure* of G is the subgroup $G_* = \langle v * G : v \in \mathcal{T} \rangle$ of W. The *closure* of G is its topological closure $\widehat{G}$ as a subgroup of W; one has

$$\widehat{G} = \left\{ g \in W \;\middle|\; \text{for all } n \in \mathbb{N} \text{ there are } g^{(n)} \in G \text{ with } g_n^{(n)} = g_n \right\}.$$

The topological closure of a layered group is always W.

The pointwise stabilizer in G of the nth layer of $\mathcal{T}$ is written $\mathsf{Stab}_G(n)$. Clearly $G/\mathsf{Stab}_G(n)$ is the permutation group on X^n generated by $\{ g_n \,|\, g \in G \}$.

Note that in a weakly branch group all rigid stabilizers are actually infinite.

Clearly "layered" implies "regular branch", which implies "branch" and "weakly regular branch", each of which imply "weakly branch".

The *finitary group* $F < W$ is defined as the layered closure of $\mathsf{Sym}(X)$. It is a locally finite, countable subgroup of W, and hence is a minimal layered subgroup. Every layered group contains F.

Fix an isometry $g \in W$. The *activity* of g at $v \in \mathcal{T}$ is $\sigma_{g@v} \in \mathsf{Sym}(X)$. The *portrait* of g is the activity map $\mathcal{T} \to \mathsf{Sym}(X)$, $v \mapsto \sigma_{g@v}$; there is a bijection between W and the set of portraits.

Consider finite-state automata with input alphabet X and states labelled by $\mathsf{Sym}(X)$. The set of portraits defined by such automata defines the group R of *finite-state isometries* of $\mathcal{T}$. Our main examples of groups, and their automorphism groups, are subgroups of R.

For a subgroup A of $\mathsf{Sym}(X)$, consider the subgroup $\widehat{A}_*$ of W consisting of those elements whose activity at any vertex is in A. If A is transitive on X, the resulting group $\widehat{A}_*$ is layered. In particular, if $A = \langle (\mathbb{1}, \mathbb{2}, \ldots, \mathbb{d}) \rangle$ and d is prime, then $C = \widehat{A}_*$ is a pro-d-Sylow of W. It consists of all tree isometries whose action below any vertex belongs to A.

Lemma 3.3. *Let G be any group acting on $\mathcal{T} = X^*$. Then the set of pre-layered subgroups of G forms a lattice of groups, closed under taking verbal subgroups.*

Proof. If H, K are pre-layered subgroups of G, then for any $g \in H \cap K$ we have $x * g \in H \cap K$, and for any $g = h_1 k_1 \ldots h_n k_n \in \langle H, K \rangle$ we have $x * g \in \langle H, K \rangle$ by Lemma 3.1.

Finally, for any $W \in F(X_1, \ldots, X_n)$ we set $W(H) = \{ W(h_1, \ldots, h_n) \,|\, h_i \in H \}$. Given $g = W(h_1, \ldots, h_n) \in W(H)$, we have $x * g = W(x * h_1, \ldots, x * h_n) \in W(H)$ so $W(H)$ is pre-layered. $\qquad\square$

Lemma 3.4. *If L is a layered group acting on $\mathcal{T} = X^*$, then there exists a transitive subgroup A of $\mathsf{Sym}(X)$ such that $L \cong L \wr A$.*

Proof. Let A be the quotient of L obtained by restricting the action of L to $X \subset X^*$. Since $x * L < L$ for all $x \in X$, and $\mathsf{Stab}_L(1) \leq L^X$, we have $1 \to \mathsf{Stab}_L(1) = L^X \to L \to A \to 1$. Furthermore, every $a \in A$ has a lift $(\ell_1, \ldots, \ell_d)a \in L$, and since $(\ell_1, \ldots, \ell_d) \in L$ we have $(1, \ldots, 1)a \in L$, defining a splitting of A in L. $\qquad\square$

Proposition 3.5 (R. Möller [26], Proposition 4). *If L is layered and $K \triangleleft L$ is a non-trivial normal subgroup, then K contains $\mathsf{Stab}_L(n)'$ for some $n \in \mathbb{N}$.*

Therefore every quotient of L is abelian by finite.

Assume now that G is generated by a symmetric set S. This induces a *metric* on G, defined by

$$(2) \qquad\qquad \|g\| = \min\{n \,|\, g = s_1 \ldots s_n \text{ with } s_i \in S\}.$$

The group G is *contracting* if there exist constants $\eta < 1$ and K such that $\|g@x\| \le \eta\|g\| + K$ for all $g \in \mathsf{Stab}_G(1)$ and $x \in X$.[3]

3.1. Automorphisms and isometries.
The following lemma is folklore:

Lemma 3.6. *Let G be a weakly branch group. Then the centralizer $\mathsf{Cent}_W(G)$ is trivial, and all conjugacy classes of G, except $\{1\}$, are infinite.*

Proof. Let $a \in W$ be non-trivial; then a moves a point $v \in \mathcal{T}$. Choose now any non-trivial $g \in \mathsf{Rist}_G(v)$; then $[a,g] \ne 1$, since it acts similarly to g on $v\mathcal{T}$.

Consider now a non-trivial $g \in G$. It moves a point $v \in \mathcal{T}$, hence its centralizer $\mathsf{Cent}_G(g)$ intersects trivially $\mathsf{Rist}_G(v)$, which is infinite, so $\mathsf{Cent}_G(g)$ has infinite index in G and g^G is infinite. $\qquad\square$

In [21], Y. Lavreniuk and V. Nekrashevych prove

Theorem 3.7. *Let G act on $\mathcal{T}$, and suppose G is weakly branch. Then*

$$\mathsf{Norm}_{\mathsf{Homeo}(\partial\mathcal{T})}(G) = \mathsf{Aut}(G).$$

Assume moreover that G is saturated. Then

$$\mathsf{Norm}_W(G) = \mathsf{Aut}(G).$$

Sketch of proof. The first step is to show that for any $\phi \in \mathsf{Aut}(G)$ and $v \in \mathcal{T}$ there is $w \in \mathcal{T}$ with $\mathsf{Rist}_G(v)^\phi \ge \mathsf{Rist}_G(w)$.

Taking the shortest such w induces a continuous map on the boundary $\partial\mathcal{T}$, which sends v to w.

If G is saturated, then $|w| = |v|$, so this map is at each level of the tree a permutation of its vertices. Since W is compact, there is a convergent subsequence of these permutations, which converges to a tree isometry. $\qquad\square$

The proof is in fact essentially model-theoretic; see [31]. Indeed it amounts to showing that the algebraic structure of a saturated weakly branch group is sufficient to reconstruct its action on $\mathcal{T}$.

Lemma 3.8. *Let G be a layered group acting on the binary tree. Then G is saturated.*

Proof. Define inductively $H_0 = G$ and $H_{n+1} = \mho_2(H_n)$, the subgroup of H_n generated by the squares of its elements. Then clearly H_n is characteristic in G for all n, and since G is layered, H_n contains for all $g \in G$ the element $\left(((\ldots(g,1)^\sigma,1)^\sigma,\ldots,1)^\sigma\right)^{2^n} = (g,\ldots,g)$ with 2^n copies of g; therefore H_n fixes X^n and acts transitively on every subtree at level n. $\qquad\square$

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[3]A more general notion is usually used [3], namely that there exists $\eta < 1$, $n \in \mathbb{N}$ and K such that $\|g@u\| \le \eta^n\|g\| + K$ for all $g \in G$ and $u \in X^n$. This notion becomes then independent of the choice of the generating set S, and there is furthermore an optimal η, which does not depend on S either.

Lemma 3.9. *Let G be a group acting on the binary tree. If G contains a weakly branch group, then G is weakly branch.*

If G contains a subgroup H such that $(\mho_2)^n(H)$ acts transitively on every subtree at level n, for all $n \in \mathbb{N}$, then G is saturated.

Proof. The first statement follows directly from the definition: if G contains the weakly branch subgroup H, then $(v * H) \cap H \neq 1$ for all $v \in \mathcal{T}$, and hence $(v * G) \cap G \neq 1$ for all $v \in \mathcal{T}$.

For the second statement, $(\mho_2)^n(G)$ fixes X^n, and also acts transitively on every subtree at level n. $\qquad\square$

Proposition 3.10. *The groups F and W are layered and saturated.*

Proof. Branchness is clear, since we have $\mathsf{Rist}_W(v) = v * W$ and $\mathsf{Rist}_F(v) = v * F$.

To prove saturatedness, we invoke Theorem 3.11; indeed choose $x \in X$ and write $W = \mathsf{Rist}_W(x) \wr_X \mathsf{Sym}(X)$; then $H_1 \lhd W$ may be chosen to be its base group W^X. Inductively, let K be the copy of H_{n-1} inside $\mathsf{Rist}_W(x) = x * W$; then $H_n \lhd W$ may be chosen as K^X, characteristic in W^X which is itself characteristic in W.

The same argument applies to F. $\qquad\square$

Wreath products exhibit a form of rigidity which was already noted independently by a number of authors:

Theorem 3.11 (Peter Neumann, [27]; Yurii Bodnarchuk, [5]; Paul Lentoudis and Jacques Tits, [22])**.** *If A, B are non-trivial groups with B acting on a set X, then the base group A^X is characteristic in the wreath product $A \wr_X B$, unless $A = \mathbb{Z}/2$ and B has an abelian subgroup B_0 of index 2 containing unique square roots of its elements.*

Furthermore, in some specific examples, a strengthening of this rigidity has been obtained: in essence, not only are all group isomorphisms induced by a tree isomorphism, but moreover there is a unique minimal tree carrying the group's action. R. Grigorchuk and J. S. Wilson showed that if G is a branch group acting on a group $\mathcal{T}$ satisfying two technical conditions, and if G also acts on another tree $\mathcal{T}'$ as a branch group, then the actions on $\mathcal{T}$ and $\mathcal{T}'$ are intertwined by map $\mathcal{T}' \to \mathcal{T}$ defined by the erasing of some levels in $\mathcal{T}'$.

In contrast, finite groups behave in a manner very different from the groups we are interested in:

Theorem 3.12. *If the finite group G acts on $\mathcal{T}$, then $\mathsf{Cent}_W(G)$ is uncountable.*

Proof. For simplicity we prove the result only for $\mathcal{T}$ the binary tree $\{\mathbb{1}, \mathbb{2}\}^*$. We proceed by induction, first on $|G|$, and then on the smallest level at which a non-trivial activity occurs, i.e. the first k such that G is not a subset of $\mathsf{Stab}_W(k)$.

If $|G| = 1$, then $\mathsf{Cent}_W(G) = W$ and we are done. If $G \leq \mathsf{Stab}_W(1)$, then consider the projections $H_\mathbb{1}, H_\mathbb{2}$ of G defined by restriction to the respective subtrees rooted at level 1. These are finite groups, of size at most $|G|$, and at least one of $H_\mathbb{1}, H_\mathbb{2}$, say $H_\mathbb{1}$, has already been covered by induction; therefore $\mathsf{Cent}_W(G)$ contains $\mathsf{Cent}_W(H_\mathbb{1}) \times \{1\}$ and is uncountable.

We may therefore assume that G contains an element $g = (g_\mathbb{1}, g_\mathbb{2})\sigma$; let also H be the projection of $\mathsf{Stab}_G(1)$ on the subtree rooted at $\mathbb{1}$. Set $Z = \mathsf{Cent}_W(H)$; by induction, Z is uncountable. Furthermore, $\mathsf{Stab}_G(1) \leq H \times H^{g_\mathbb{2}^{-1}}$, so the centralizer of $\mathsf{Stab}_G(1)$ contains $Z \times Z^{g_\mathbb{2}^{-1}}$.

The centralizer of g contains all elements $(z, z^{g_\mathbb{2}^{-1}})$ with $z \in \mathsf{Cent}_W(g_\mathbb{1} g_\mathbb{2})$; and since $g^2 = (g_\mathbb{1} g_\mathbb{2}, g_\mathbb{2} g_\mathbb{1}) \in \mathsf{Stab}_G(1)$, we see that $\mathsf{Cent}_W(G)$ contains all $(z, z^{g_\mathbb{2}^{-1}})$ for $z \in Z$; therefore $\mathsf{Cent}_W(G)$ is uncountable. $\qquad\square$

Consider also the following example of a group with uncountable normalizer: let U be an infinite minimal connecting set of vertices of the tree $\mathcal{T}$, i.e. a subset of $\mathcal{T}$'s vertices that intersects any infinite ray. For every $u \in U$, let $H(u)$ be a subgroup of $\mathsf{Aut}(\mathcal{T})$. Let G be the group generated by all $u * H(u)$, with u ranging over U. Then G is isomorphic to the direct sum of the $H(u)$'s, and the normalizer of G in W contains the cartesian product of the $H(u)$'s.

Let us note finally a result related to Theorem 3.7, and expressed in terms of commensurators: an *almost automorphism* of G is an automorphism between two finite-index subgroups of G. Two almost automorphisms are equivalent if they agree on on a finite-index subgroup of G. The set of equivalence classes of almost automorphisms carries a natural group structure, and is called the *abstract commensurator* of G.

On the other hand, if G is a subgroup of K, then the commensurator of G in K is $\left\{ k \in K \,\middle|\, [G : G \cap G^k] < \infty, [G^k : G \cap G^k] < \infty \right\}$.

Theorem 3.13 (Claas Röver, [30]). *The abstract commensurator of a weakly branch group G acting on T is isomorphic to the commensurator of G in $\mathsf{Homeo}(\partial \mathcal{T})$.*

3.2. **The automorphism tower.** Our first purpose is, given a suitable G, to identify a group G_* that is much smaller than W (i.e., for instance, is countable); but still is large enough to contain the whole automorphism tower of G. This could be interpreted as a strengthening of Theorem 3.7.

For any $n \in \mathbb{N}$ set $G_n = \langle v * G : v \in \mathcal{T}, |v| \leq n \rangle$, and $G_* = \bigcup_{n \geq 0} G_n$. The following is immediate:

Lemma 3.14. *Assume G is weakly branch and recurrent. Then G_n is weakly branch for all n, and the G_n form an ascending tower. Its limit G_* is layered, and is the smallest layered group containing G. If G is countable, then so is G_*.*

Theorem 3.15. *If G is a finitely generated group with a recurrent, saturated, and weakly branch action on $\mathcal{T}$, and if G satisfies $G_* = \bigcup_{n \in \mathbb{N}} \mathsf{Norm}^n(G)$, then G_* is complete; in other words, G has height at most ω.*

We remark that the condition $G_* = \bigcup_{n \in \mathbb{N}} \mathsf{Norm}^n(G)$ implies that for all $n \in \mathbb{N}$ there are $k, \ell \in \mathbb{N}$ such that $\times^k G < \mathsf{Norm}^n(G) < (\times^\ell G)(\wr^\ell C_2)$, since G is finitely generated.

Proof. Take $\phi \in \mathsf{Aut}(G_*)$. Then by Theorem 3.7 we have $\phi \in \mathsf{Aut}\,\mathcal{T}$, and similarly $\psi = \phi^{-1} \in \mathsf{Aut}\,\mathcal{T}$. Since by definition G is finitely generated, there exists $n \in \mathbb{N}$ such that $\phi(G) \leq \mathsf{Norm}^n(G)$ and $\psi(G) \leq \mathsf{Norm}^n(G)$.

Since $G_* = \bigcup_{m \in \mathbb{N}} G_m$, there is also an $m \in \mathbb{N}$ such that $\phi(G) \leq G_m$ and $\psi(G) \leq G_m$.

Let $u, \ldots, v$ be all the vertices on level m of $\mathcal{T}$, and let us decompose $\phi = (\phi_u, \ldots, \phi_v)\sigma$ and $\psi = (\psi_u, \ldots, \psi_v)\sigma^{-1}$. Since G is recurrent, for any $g \in G$ there exists a $g' \in G$ with level-m decomposition $(g, *, \ldots, *)$. Then $(g')^\phi = (g^{\phi_u}, *, \ldots, *)^\sigma$ belongs to G, and therefore ϕ_u induces by conjugation an endomorphism of G.

Then $(g, *, \ldots, *) = g' = \phi\psi(g') = (\phi_u(*), *, \ldots, *)$, and therefore $\phi_u(G) = G$, so ϕ_u induces an automorphism of G. We have shown $\mathsf{Aut}(G_*) \leq (\mathsf{Aut}(G)F)_* = G_*$.   □

4. Layered groups

We describe in this section the automorphism tower of a layered group L acting on the binary tree $\mathcal{T} = X^*$, with $X = \{\mathbb{1}, \mathbb{2}\}$; recall that such group has a decomposition $L = (L \times L) \rtimes \langle \sigma \rangle$, where σ denotes the transposition of the top two branches of the binary tree.

Denote $\mathsf{Aut}(L)$ by A. By Lemma 3.8 and Theorem 3.7, every automorphism α of L acts on the binary tree, and therefore we have $A \leq (W \times W) \rtimes \langle \sigma \rangle$. Then, given $\alpha = (\alpha_{\mathbb{1}}, \alpha_2)\sigma^i \in A$, we have $(\ell, 1)^\alpha = (\ell^{\alpha_{\mathbb{1}}}, 1)^{\sigma^i} \in L$, and similarly $(1, \ell)^\alpha \in L$ so $\alpha_i \in A$ for all $i \in X$ and $A \leq (A \times A) \rtimes \langle \sigma \rangle$. Furthermore, $\sigma^\alpha = \sigma^{1-i}(\alpha_2^{-1}\alpha\mathbb{1}, \alpha_{\mathbb{1}}^{-1}\alpha_2)\sigma^i \in L$, so $\alpha_2^{-1}\alpha_{\mathbb{1}} \in L$ and $A \leq (A \times 1)L$. In other words, α can be written as a (possibly infinite) product

$$\alpha = \cdots (\mathbb{1}^n * \ell_n) \cdots (\mathbb{1} * \ell_1)\ell_0,$$

for some $\ell_i \in L$. On the other hand, such a product defines an endomorphism (but not necessarily an automorphism) of L, by conjugation.

Note also that α is finite state provided the sequence (ℓ_n) is ultimately periodic (see [7]). That same paper shows that, for F the finitary group, $\mathsf{Aut}(F)$ contains a copy of W and is therefore uncountable.

We mention in passing another proof that $\mathsf{Aut}(F)$ is uncountable: recall that an automorphism ϕ of a group G is *locally inner* if $\phi_{|X}$ is inner for every finite $X \subset G$. Let $\mathsf{Linn}(G)$ denote the group of locally inner automorphisms of G.

Proposition 4.1 ([18], page 37)**.** *Let G be a countably infinite, locally finite group, and suppose that $\mathsf{Cent}_G(F)$ is not central in G, for all finite subgroups F of G. Then $|\mathsf{Linn}(G)| = 2^{\aleph_0}$.*

It is, however, unknown whether $|\mathsf{Aut}(G)| = 2^{\aleph_0}$ for all countably infinite, locally finite groups G.

Definition 4.2. For a layered group L, we write $\mathsf{Aut}^i(L)$ for the i-th term in the automorphism tower of L. For groups $K \lhd G$, we write

$$\Omega(G, K) = \{v \in G | v^2 \in K \text{ and } [G, v] \leq K\}.$$

We shall be concerned, in this section, with groups $G \leq W$ that admit a decomposition

(3) $$G = (\times U)\langle \sigma \rangle(\triangle S)$$

where $U \lhd S$ is maximal, and G contains a fixed layered group L. By Theorem 3.7 we have $\mathsf{Aut}\, G = \mathsf{Norm}_W(G) =: \mathsf{Norm}\, G$. The form (3) is preserved by some important operations, namely if $G = (\times U)\langle \sigma \rangle(\triangle S)$ and $H = (\times V)\langle \sigma \rangle(\triangle T)$, then

$$G \cap H = (\times(U \cap V))\langle \sigma \rangle(\triangle(S \cap T)),$$

(Lemma 4.6) $$\Omega(G, H) = (\times(U \cap \Omega(S, V)))\langle \sigma \rangle(\triangle(\Omega(S, T))),$$

(Theorem 4.4) $$\mathsf{Norm}(G) = (\times\Omega(S, U))\langle \sigma \rangle(\triangle(\mathsf{Norm}\, U \cap \mathsf{Norm}\, S)).$$

Lemma 4.3. *Consider a group $H = (\times K)\langle \sigma \rangle$. Then*

$$\mathsf{Norm}(H) = (\times K)\langle \sigma \rangle(\triangle\,\mathsf{Norm}(K)),$$

where $\times K$ is the largest geometrically decomposable subgroup of $\mathsf{Norm}(H)$ and $\triangle\,\mathsf{Norm}(K)$ is the largest diagonal subgroup of $\mathsf{Norm}(H)$.

Proof. Choose $\alpha = (\alpha_{\mathbb{1}}, \alpha_2)\sigma^i \in \mathsf{Norm}(H)$. Then $(\alpha_{\mathbb{1}}, \alpha_2) \in \mathsf{Norm}(H)$, and $\alpha_{\mathbb{1}}, \alpha_2 \in \mathsf{Norm}(K)$. Thus, $\alpha \in \times\,\mathsf{Norm}(K)\langle \sigma \rangle$ and $[(\alpha_{\mathbb{1}}, \alpha_2), \sigma] = (\alpha_{\mathbb{1}}^{-1}\alpha_2, \alpha_2^{-1}\alpha_{\mathbb{1}}) \in \times K$. Therefore, $\alpha_2 = \alpha_{\mathbb{1}}k$ for some $k \in K$; that is, $\alpha = (\alpha_{\mathbb{1}}, \alpha_{\mathbb{1}}k)\sigma^i$. Since $\triangle\alpha_{\mathbb{1}} \in \mathsf{Norm}(H)$ for all $\alpha_{\mathbb{1}} \in \mathsf{Norm}(K)$, the decomposition $\mathsf{Norm}(H) = (\times K)\langle \sigma \rangle(\triangle\,\mathsf{Norm}(K))$ follows.

The rest of the assertions are easy. $\square$

Theorem 4.4. *Consider $R = (\times U)\langle \sigma \rangle(\triangle S)$ where U is a normal subgroup of S. Then*

$$\mathsf{Norm}(R) = (\times V)\langle \sigma \rangle(\triangle T)$$

where $V = \Omega(S, U)$ and $T = \mathsf{Norm}(U) \cap \mathsf{Norm}(S)$.

Furthermore, V is a normal subgroup of T, and $[\times V, R] \leq (\times U)(\triangle V)$, and the quotient group V/U is a T/S-module, and

$$\mathsf{Norm}(R)/R \cong (V/U)(T/S).$$

Proof. First, it is clear that V is a normal subgroup of T and that $\times V$ normalizes both $\times U$ and $\triangle S$.

Choose $(v,1) \in \times V$; then $[(v,1),\sigma] = (v^{-1},v) = (v^{-2},1)(v,v)$ where by the definition of V we have $v^{-2} \in U$ and $(v,v) \in \triangle S$. Therefore $\times V \leq \mathsf{Norm}(R)$. Easily, $\triangle T \leq \mathsf{Norm}(R)$.

Choose next $\beta \in \mathsf{Norm}(R)$. Then we may assume $\beta = (\beta_\mathbb{1}, \beta_\mathbb{2})$. As $\times U$ is normal in R, we conclude that $\beta_\mathbb{1}, \beta_\mathbb{2} \in \mathsf{Norm}(U)$ and commutation with $\triangle S$ shows that $\beta_\mathbb{1}, \beta_\mathbb{2} \in \mathsf{Norm}(US) = \mathsf{Norm}(S)$. Thus, $\beta_\mathbb{1}, \beta_\mathbb{2} \in T$ and

$$\mathsf{Norm}(R) \leq (\times T)\langle \sigma \rangle.$$

Now, $\sigma^\beta = (\beta_\mathbb{1}^{-1}\beta_\mathbb{2}, \beta_\mathbb{2}^{-1}\beta_\mathbb{1})\sigma = (u_\mathbb{1}s, u_\mathbb{2}s)\sigma$ for some $u_\mathbb{1}, u_\mathbb{2} \in U, s \in S$. Since $\sigma^2 = 1$, we have $u_\mathbb{2}s = (u_\mathbb{1}s)^{-1} = u_\mathbb{1}^{-s}s^{-1}$, and therefore

$$s^2 = u_\mathbb{2}^{-1}u_\mathbb{1}^{-s} \in U, \qquad \beta_\mathbb{2} = \beta_\mathbb{1}u_\mathbb{1}s.$$

Thus, $\beta = (\triangle\beta_\mathbb{1})(1, u_\mathbb{1}s)$ with $\beta_\mathbb{1} \in T$. Since we have already shown $\triangle T \leq \mathsf{Norm}(R)$, and as $1 \times U \leq R$, we may assume $\beta = (1, u_\mathbb{1}s)$, and as $1 \times U \leq R$ we may assume further that $\beta = (1,s)$. Now, $[\triangle S, \beta] = 1 \times [S,s] \leq R$ and therefore $[S,s] \leq U$. Hence, $s \in \Omega(S,U) = V$ and $\mathsf{Norm}(R) = (\times V)\langle\sigma\rangle(\triangle T)$.

We then have

$$[\times V, R] = [\times V, \times U][\times V, \triangle S\langle\sigma\rangle]$$
$$= (\times[V,U])(\times[V,S])\{(v^{-1},v) \mid v \in V\}$$
$$\leq (\times U)(\triangle V),$$

since $[V,S], (V^2, 1) \leq U$. We observe the following facts:

$$(\triangle T) \cap R = (\triangle T) \cap (\triangle(US)) = \triangle(T \cap S) = \triangle S,$$
$$((\times V)\langle\sigma\rangle) \cap R = (\times U)(\triangle V)\langle\sigma\rangle,$$

V/U is a T-module and the kernel of the action contains S.

Finally, let W be a transversal of U in V and X a transversal of S in T. Then $(W \times 1)(\triangle X)$ is a transversal of R in $\mathsf{Norm}(R)$, and hence $\mathsf{Norm}(R)/R \cong (V/U)(T/S)$. $\qquad \square$

Remark 4.5. Suppose $\mathsf{Norm}(S) = S$. Then

$$\mathsf{Norm}^i(R) = (\times U_i)\langle\sigma\rangle(\triangle S)$$

where $U_0 = U$ and $U_i = \Omega(S, U_{i-1})$. The subgroups U_i form a hypercentral series of S with respect to U.

In particular, if S is finitely generated, then $\mathsf{Norm}^\omega(R) = \mathsf{Norm}^{\omega+1}(R)$, because letting $U_\omega = \bigcup_{n \in \mathbb{N}} U_n$ we have $\Omega(S, U_\omega) = U_\omega$.

Lemma 4.6. *Consider $G = (\times U)\langle\sigma\rangle(\triangle S)$, and let $H = (\times V)\langle\sigma\rangle(\triangle T)$ be a normal subgroup of G. Then*

$$\Omega(G, H) = (\times(U \cap \Omega(S,V)))\langle\sigma\rangle(\triangle\Omega(S,T)).$$

Proof. By assumption we have $V \triangleleft S, T \triangleleft S, U \triangleleft S, V \triangleleft T$, and $U \leq \Omega(T, V)$.

Set $Q = \Omega(G, H)$. First, consider $y = (x,1)$ with $x \in U \cap \Omega(S,V)$. Then $x^2 \in V$ and $[x,S] \subseteq V$ so $y^2 \in H$ and $[y,G] \in H$, so $y \in Q$. Conversely, if $y = (1,x) \in Q$, then $[y,G] \in H, [y,\sigma] \in H$ and $y^2 \in H$ imply $x \in U \cap \Omega(S,V)$.

$\mathsf{Norm}(S_3)$ $\mathsf{Norm}(U_3)$ U_6

S_4 Ω

$N_2 = \mathsf{Norm}(S_2)$ $\mathsf{Norm}(U_2)$ U_5

S_3 Ω

U_4

$N_1 = S_1 = S_2$ Ω

U_3
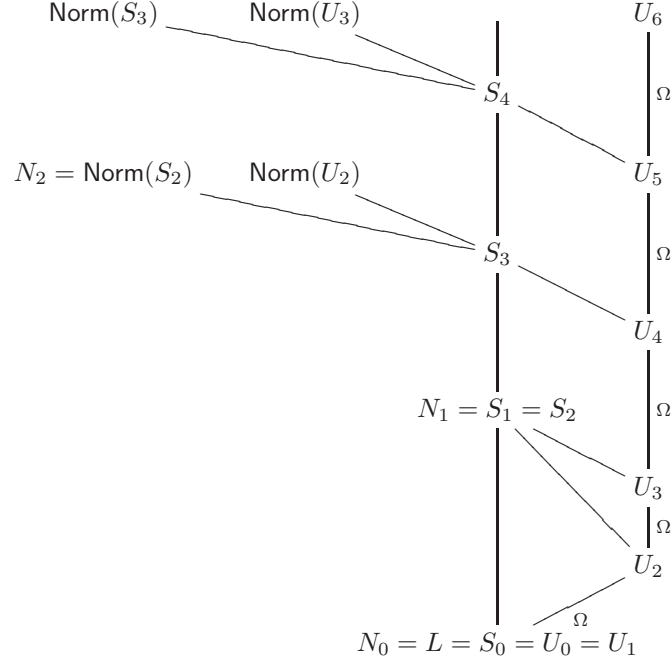
Ω

U_2

Ω

$N_0 = L = S_0 = U_0 = U_1$

FIGURE 1. Some groups appearing in Theorem 4.7

Consider next $y = (x, x)$ with $x \in \Omega(S, T)$. Then $x^2 \in T$ and $[x, S] \subseteq T$ so $y^2 \in H$ and $[y, G] \in H$, so $y \in Q$. Conversely, if $y = (x, x) \in Q$, then $[y, G] \in H$ and $y^2 \in H$ imply $x \in \Omega(S, T)$. $\qquad\square$

Theorem 4.7. *Let L be a layered group, $L = (\times L)\langle\sigma\rangle$ and let $(N_i = \mathsf{Norm}^i(L))_{i \geq 0}$ be the normalizer tower of L. Then we have the equations (see Figure 1)*

$$N_0 = U_0 = S_0 = L,$$
$$N_i = (\times U_i)\langle\sigma\rangle(\triangle S_i),$$
$$U_i = \Omega(S_{i-1}, U_{i-1}),$$
$$S_i = \mathsf{Norm}(U_{i-1}) \cap \mathsf{Norm}(S_{i-1}).$$

For $1 \leq i \leq 4$, the description of N_i by the pair (U_i, S_i) is as follows:

$$U_1 = (\times L)\langle\sigma\rangle = L, \qquad\qquad S_1 = N_1;$$
$$U_2 = (\times L)\langle\sigma\rangle(\triangle U_2), \qquad\qquad S_2 = N_1;$$
$$U_3 = (\times L)\langle\sigma\rangle(\triangle U_3), \qquad\qquad S_3 = (\times U_2)\langle\sigma\rangle(\triangle N_1) = N_2;$$
$$U_4 = (\times U_2)\langle\sigma\rangle(\triangle\Omega(N_1, U_3)), \qquad\qquad S_4 = (\times\Omega(U_3, U_1))\langle\sigma\rangle(\triangle N_1).$$

Furthermore, the outer-normalizer groups are

$$N_2/N_1 \cong U_2/U_1,$$
$$N_3/N_2 \cong (U_3/U_2)(N_2/N_1),$$
$$N_4/N_3 \cong (U_4/U_3)(\Omega(U_3, L)/U_2)(N_2/N_1),$$

where U_{i+1}/U_i and $\Omega(U_3, L)/U_2$ are elementary abelian 2-groups.

Proof. Since $L = (\times L)\langle\sigma\rangle(\triangle L)$, the general form of the normalizer subgroups $\mathsf{Norm}^i(L) = (\times U_i)\langle\sigma\rangle(\triangle S_i)$ is determined by Theorem 4.4. We compute for $i \leq 4$:

i = 1. We have $N_1 = (\times U_1)\langle\sigma\rangle(\triangle S_1)$, with $U_1 = \Omega(L, L) = L = U_0$ and $S_1 = \mathsf{Norm}(L) \cap \mathsf{Norm}(L) = \mathsf{Norm}(L) = N_1$. We do not obtain any information about the outer-normalizer group N_1/N_0, since

$$N_1/N_0 \cong (U_1/U_0)(S_1/S_0) = S_1/S_0 \cong N_1/N_0.$$

i = 2. First, $N_2 = (\times U_2)\langle\sigma\rangle(\triangle S_2)$, with $U_2 = \Omega(N_1, U_1)$ and $S_2 = \mathsf{Norm}(U_1) \cap \mathsf{Norm}(S_1)$.
    Secondly, $S_2 = \mathsf{Norm}(L) \cap \mathsf{Norm}(N_1) = N_1 \cap N_2 = N_1$.
    Thirdly, $N_0 = L = L(\triangle L) \leq U_2 = \Omega(N_1, U_1) \leq N_1 = L(\triangle N_1)$. By Lemma 4.6, $U_2 = L(\triangle W)$, with $W = \Omega(N_1, N_0) = \Omega(N_1, U_1) = U_2$; therefore,

$$U_2 = L(\triangle U_2) = U_1(\triangle U_2).$$

    Fourthly, $\mathsf{Norm}(U_2) = (\times V)\langle\sigma\rangle(\triangle Y)$, with $V = \Omega(U_2, U_1) = U_2$ and $Y = \mathsf{Norm}(U_1) \cap \mathsf{Norm}(U_2) = N_1 \cap \mathsf{Norm}(U_2) = N_1$, and therefore $N_2 = \mathsf{Norm}(U_2)$.
    Finally, $N_2/N_1 \cong (U_2/U_1)(S_2/S_1) = U_2/U_1$, an elementary abelian 2-group.

i = 3. First, $N_3 = (\times U_3)\langle\sigma\rangle(\triangle S_3)$, with $U_3 = \Omega(S_2, U_2)$ and $S_3 = \mathsf{Norm}(U_2) \cap \mathsf{Norm}(S_2)$.
    Secondly, $S_3 = N_2 \cap \mathsf{Norm}(S_2) = N_2$.
    Thirdly, $U_3 = \Omega(N_1, U_2)$ and $L(\triangle U_2) \leq U_3 \leq N_1 = L(\triangle N_1)$, so $U_3 = L(\triangle W)$ where $W = \Omega(N_1, U_2) = U_3$; and hence, $U_3 = L(\triangle U_3)$.
    Finally, $N_3/N_2 \cong (U_3/U_2)(S_3/S_2) = (U_3/U_2)(N_2/N_1)$ where, as we had shown, $N_2/N_1 \cong U_2/U_1$.

i = 4. First, $N_4 = (\times U_4)\langle\sigma\rangle(\triangle S_4)$, with $U_4 = \Omega(S_3, U_3) = \Omega(N_2, U_3)$ and $S_4 = \mathsf{Norm}(U_3) \cap \mathsf{Norm}(S_3)$.
    Secondly, $S_4 = \mathsf{Norm}(U_3) \cap \mathsf{Norm}(N_2) = \mathsf{Norm}(U_3) \cap N_3$.
    Thirdly, from $U_3 = (\times L)\langle\sigma\rangle(\triangle U_3)$ we find

$$\mathsf{Norm}(U_3) = (\times W)\langle\sigma\rangle(\triangle(\mathsf{Norm}(L) \cap \mathsf{Norm}(U_3))),$$

with $W = \Omega(U_3, L)$. Then $\mathsf{Norm}(L) \cap \mathsf{Norm}(U_3) = N_2 \cap \mathsf{Norm}(U_3) = S_3 \cap \mathsf{Norm}(U_3) = N_2$, and $\mathsf{Norm}(U_3) = (\times W)\langle\sigma\rangle(\triangle N_2)$. Since $W = \Omega(U_3, L) \leq U_3 = \Omega(N_1, L)$, we conclude $\mathsf{Norm}(U_3) \leq N_3$; and hence, $S_4 = \mathsf{Norm}(U_3)$.
    Fourthly, $U_4 = \Omega(N_2, U_3)$. Since

$$U_3 = (\times L)\langle\sigma\rangle(\triangle U_3) \leq U_4 \leq N_2 = (\times U_2)\langle\sigma\rangle(\triangle N_1)$$

and $(\times U_2) \leq \Omega(N_2, U_2) \leq \Omega(N_2, U_3)$, we have $(\times U_2)\langle\sigma\rangle(\triangle U_3) \leq U_4 \leq N_2 = (\times U_2)\langle\sigma\rangle(\triangle N_1)$. As in the previous case, $U_4 = (\times U_2)\langle\sigma\rangle(\triangle V)$, with $V = \Omega(N_1, U_3)$.
    Finally, $N_4/N_3 \cong (U_4/U_3)(S_4/S_3)$, and

$$S_4/S_3 = \mathsf{Norm}(U_3)/N_2 \cong ((\times W)\langle\sigma\rangle(\triangle N_2))/((\times U_2)\langle\sigma\rangle(\triangle N_1)) \cong (W/U_2)(N_2/N_1).$$

$\square$

It was shown in [34] that, up to conjugation, the maximal 2-subgroups M of $\mathsf{Aut}(\mathcal{T})$ can be of two types: first, subgroups of the form $M = M_1 \times M_2$ where M_1, M_2 are two nonconjugate maximal 2-subgroups of $\mathsf{Aut}(\mathcal{T})$. The second type consists of layered groups $M = (M \times M)\langle\sigma\rangle$.

Corollary 4.8. *Suppose L is a maximal 2-subgroup of $\mathsf{Aut}(\mathcal{T})$ which is layered. Then $N_2(L) = N_1(L)$, i.e. L has height at most 2.*

We now proceed towards the proof of Theorem 1.3.

Lemma 4.9. *Let L be a layered group. Then for every $i \in \mathbb{N}$ we have $\mathsf{Rist}_{\mathsf{Aut}^i(L)}(w) = L$ for all $w \in \mathcal{T}$ with $|w| \geq i$.*

The subgroup $\times^i L$ is characteristic in $\mathsf{Aut}^i(L)$.

Proof. For $i = 0$ the assertions are clear, while for $i = 1$ they follow from Lemma 4.3. We then prove both claims simultaneously by induction.

Assume that $\mathsf{Rist}_{\mathsf{Aut}^i(L)}(w) = L$ for all $w \in \mathcal{T}$ with $|w| \geq i$, and $\times^i L \lessdot \mathsf{Aut}^i(L)$. It follows from Theorem 3.7 that $\times^{i+1} L$ is characteristic in $\mathsf{Aut}^i(L)$, and from Lemma 4.3 we have $\mathsf{Rist}_{\mathsf{Aut}^{i+1}(L)}(w) \leq \mathsf{Aut}(L)$ as soon as $|w| \geq i$. Therefore $\mathsf{Rist}_{\mathsf{Aut}^{i+1}(L)}(wx) \leq \mathsf{Rist}_{\mathsf{Aut}(L)}(x) \leq L$ for all $x \in \mathcal{T}$, and $\mathsf{Rist}_{\mathsf{Aut}^{i+1}(L)}(w) = L$ as soon as $|w| \geq i + 1$.

Then, again by Theorem 3.7, $\times^{i+1} L \lessdot \mathsf{Aut}^{i+1}(L)$, because the product of all rigid stabilisers on a level of the tree is always characteristic. $\square$

Lemma 4.10. *Let L be a layered group, and set $N = \mathsf{Aut}(L)$. Then $\mathsf{Aut}^i(L) \leq N \wr (\wr^{i-1} C_2)$ for all $i \in \mathbb{N}$.*

Proof. By Theorem 3.7, the terms N_i of the automorphism tower of L are subgroups of W. By Lemma 4.9, the rigid stabilisers at level i in $\mathcal{T}$ are all L. Given $\alpha \in \mathsf{Aut}^{i+1}(L)$, we decompose it i times as $\alpha = (\alpha_u, \ldots, \alpha_v)\pi$ with $\alpha_w \in W$ for all $w \in \mathcal{T}$ of length i, and $\pi \in \wr^i C_2$.

Since α normalizes $\mathsf{Aut}^i(L)$, each α_u must normalize $\mathsf{Rist}_{\mathsf{Aut}^i(L)}(u) = L$, so $\alpha_u \in N$. $\square$

Proof of Theorem 1.3. The first part follows directly from Lemma 4.10.

For the second part, Theorem 4.4 tells us how to compute $\mathsf{Norm}^{i+1}(L)$ from $\mathsf{Norm}^i(L)$. It remains to check that, given $\mathsf{Norm}^i = (\times U_i)\langle \sigma \rangle (\triangle S_i)$ with $U_i, S_i \in \mathcal{C}$, we have U_{i+1} and $S_{i+1} \in \mathcal{C}$. This is clealy true for U_{i+1}, and for S_{i+1} the computations of Theorem 4.7 show that it holds for S_{i+1} if $i \leq 3$.

Consider then $i \geq 4$, and the group $U_i \in \mathcal{C}$. By Lemma 4.9, we may write

$$U_i = \left(\prod_{Z \in \{\times, \triangle\}^i} Z\, G_Z \right) (\wr^i \langle \sigma \rangle),$$

and we have $G_{\times^i} = L$. The recursive computation of $\mathsf{Norm}\, U_i$, following Theorem 4.4, asks us to compute $\mathsf{Norm}(G_{\times^i}) = N$, and its intersection with some of the $\mathsf{Norm}(G_Z)$; for this last step it suffices to compute $\mathsf{Norm}_N(G_Z)$ and take intersections, all operations that leave elements of $\mathcal{C}$.

The same argument applies to $\mathsf{Norm}_{U_i}(S_i) = S_{i+1}$. $\square$

We next construct within W a family of layered subgroups of arbitrary height $n \in \mathbb{N}$. Set

$$L_n = \langle \gamma_n(W), F \rangle,$$

where $\gamma_n(W)$ is the nth term in the lower central series of W.

Theorem 4.11. *L_n is a layered group of height $n - 1$; we have $\mathsf{Aut}^m(L_n) = L_{n-m}$ for all $m \in \{0, \ldots, n - 1\}$.*

Proof. We note that L_n is layered by Lemma 3.3.

We recall the following facts from [4]: the quotient $\gamma_i(W)/\gamma_{i+1}(W)$ is isomorphic to the $\mathbb{Z}/2$-module $\{f : \mathbb{N} \to \mathbb{Z}/2\}$, and maps to $\gamma_{i+1}(W)/\gamma_{i+2}(W)$ with finite kernel and co-kernel by commutation with $\dashv^k \sigma$, where $2^k || i$.

It suffices to check $\mathsf{Norm}\, L_{n+1} = L_n$ for all $n \in \mathbb{N}$. First, $[L_n, L_{n+1}] \leq L_{n+1}$ because $[\gamma_n(W), F] \leq \gamma_{n+1}(W)$ and $[\gamma_n(W), \gamma_{n+1}(W)] \leq \gamma_{n+1}(W)$; therefore $\mathsf{Norm}\, L_{n+1} \geq L_n$.

Second, given $g \notin L_n$, there exists $i < n$ with $g \in L_i \setminus L_{i+1}$; then by [4] there exists $f \in F$ with $[g, f] \notin L_{i+2}$, so in particular $[g, f] \notin L_{n+1}$ and $g \notin \mathsf{Norm}\, L_{n+1}$. $\qquad \square$

4.1. **The dihedral, affine and cardioid group.** We compute here the automorphism and normalizer tower of three examples; the first two are not layered, while the third one is. We start by fixing some common notation. Consider the adding machine $\tau = (1, \tau)\sigma$ acting on the binary tree $\mathcal{T} = \{1, 2\}^*$. It generates an infinite cyclic group. Consider also the order-2 isometry δ of $\mathcal{T}$, defined by $\delta = (\delta, \delta)\sigma$.

Theorem 4.12. *The group $H = \langle \tau, \delta \rangle$ is infinite dihedral, and satisfies $\mathsf{Norm}_W(H) = H$.*

Proof. We have $\delta = u_{-1}\tau^{-1}$, and therefore $\tau^\delta = \tau^{-1}$ and H is infinite dihedral.

Then $\mathsf{Aut}(H)$ is also dihedral, generated by H and a new element υ satisfying $\upsilon^2 = \tau$ and $\upsilon^\delta = \upsilon^{-1}$. We have $[\mathsf{Aut}(H) : H] = 2$.

However, there exists no tree isometry g satisfying $g^2 = \tau$; therefore $\mathsf{Norm}(H) = H$ and the tower of H strictly contains the normalizer tower. $\qquad \square$

Note that H is a finitely generated solvable group with $\mathsf{Norm}(H)$ countable. In contrast, if B is an abelian or finite 2-subgroup of $\mathsf{Aut}\,\mathcal{T}$, then $\mathsf{Cent}_{\mathsf{Aut}\,\mathcal{T}}(B)$ is uncountable.

Consider next the topological closure of $\langle \tau \rangle$. It is $T = \{\tau^x \mid x \in \mathbb{Z}_2\} \cong \mathbb{Z}_2$, the *triangular group*. Concretely, τ^x can be defined as

$$\tau^x = \begin{cases} (\tau^{x/2}, \tau^{x/2}) & \text{if } x \in 2\mathbb{Z}_2, \\ (\tau^{(x-1)/2}, \tau^{(x+1)/2})\sigma & \text{if } x \in 2\mathbb{Z}_2 + 1. \end{cases}$$

Consider also the group $U = \{u_x \mid x \in \mathbb{Z}_2^\bullet\} \cong \mathbb{Z}_2^\bullet$, the *unitary group*, where

$$u_x = (u_x, u_x \tau^{(x-1)/2}).$$

Set finally $A = TU$, the *affine group*. The element $\tau^x u_y$ can be represented by the matrix $\left(\begin{smallmatrix} 1 & x \\ 0 & y \end{smallmatrix}\right)$ over $\mathbb{Z}_2$.

Theorem 4.13. *We have*

$$\mathsf{Norm}_W(T) = A, \qquad \mathsf{Norm}_W(A) = A.$$

Proof. We first note that $\mathsf{Cent}_W(T) = T$. If $\alpha \in W$ fixes τ, write $\alpha = (\alpha_1, \alpha_2)\sigma^i$. Then $\tau^\alpha = \sigma^i(\alpha_1^{-1}\alpha_2, \alpha_2^{-1}\tau\alpha_1)\sigma^{1-i} = (1, \tau)\sigma$, so either $i = 0, \alpha_1 = \alpha_2 \in \mathsf{Cent}_W(\tau)$ or $i = 1, \alpha_1 = \alpha_2\tau^{-1} \in \mathsf{Cent}_W(\tau)$; so in all cases, $\alpha \in T$. Since T has $\langle \tau \rangle$ as a dense subgroup, the claim follows.

Now since W acts continuously on itself by conjugation, any $a \in \mathsf{Norm}_W(T)$ is determined by its value on τ, and a simple computation shows that $\tau^{u_x} = \tau^x$, proving the first equality.

Next, T is a $\mathsf{Norm}_W(A)$-invariant subgroup of A, since it can be defined as the set of fixed-point-free permutations of A; therefore any $a \in \mathsf{Norm}_W(A)$ is again determined by its value on τ, and the second equality follows. $\qquad \square$

However, the automorphism tower of T grows higher: even restricting to continuous automorphisms, we have $\mathsf{Aut}^n(A) = \frac{1}{2^n}\mathbb{Z}_2 \rtimes \mathbb{Z}_2^\bullet \cong A$, and $\mathsf{Aut}^\omega(A) = \mathbb{Q}_2 \rtimes \mathbb{Z}_2^\bullet$, and $\mathsf{Aut}^{\omega+1}(A) = \mathsf{Aut}^{\omega+2}(A) = \mathbb{Q}_2 \rtimes \mathbb{Q}_2^\bullet$.

Our last example is the group $G = \langle F, \tau \rangle$, where F denotes the "finitary group" generated by $\{1^n * \sigma \mid n \in \mathbb{N}\}$. We call G the *Cardioid group*; its name derives from the "monodromy construction" of V. Nekrashevych [3], when applied to the polynomial $f(z) = z^2 + \epsilon$, with ϵ in the Mandelbrot set's main cardioid. We briefly summarize its construction: let M be the Riemann surface $\mathbb{C} \setminus \overline{\{f^n(\epsilon)\}_{n \geq 0}}$, and let $M_0 = M \setminus \{0\}$ be an open subsurface of M. Then f defines a 2-fold covering from M_0 to M. Let $t \in M$ be an arbitrary point, and let

$\mathcal{T} = \bigsqcup_{n\geq 0} f^{-n}(t) \times \{n\}$ be the disjoint union of all preimages of t. It has naturally the structure of a binary tree, with edges connecting $(x, n+1)$ to $(f(x), n)$. We let $\pi_1(M, t)$ act by monodromy (path lifting) on $\mathcal{T}$, and denote by $\mathsf{IMG}(f)$ the quotient of $\pi_1(M, t)$ by the kernel of the action.

It can be shown that $G = \mathsf{IMG}(f)$. Indeed $\pi_1(M, t)$ is free, and as generators of $\pi_1(M, t)$ one may choose a loop around ∞, giving τ, and loops encircling $\overline{\{f^m(\epsilon)\}_{m\geq n}}$, giving $\mathbb{1}^n * \sigma$.

Theorem 4.14.   (1) G is a countable layered group, and satisfies $G/G' = \mathbb{Z} \oplus (\mathbb{Z}/2)^\infty$.
   (2) $\mathsf{Norm}(G) = \langle F, T\rangle D$, where $D = \langle\delta\rangle \cong \mathbb{Z}/2$.
   (3) $\mathsf{Norm}^2(G) = \mathsf{Norm}(G)$.

Proof. Clearly G is layered, because $\tau\sigma = (1, \tau) \in G$. The abelianization of W is $(\mathbb{Z}/2)^{\mathbb{N}}$, and the image of G in W/W' is generated by the finite-support and constant functions. Furthermore, the image of τ in G/G' has infinite order, since $\tau^{2^n} \in G'$ implies $\tau \in G'$ contradicting the above.

For the second assertion, we first check that T normalizes F; indeed

$$\sigma^{\tau^x} = \begin{cases} (\tau^{-x/2}, \tau^{-x/2})\sigma(\tau^{x/2}, \tau^{x/2}) & \text{if } x \in 2\mathbb{Z}_2 \\ \sigma(\tau^{(1-x)/2}, \tau^{(-1-x)/2})\sigma(\tau^{(x-1)/2}, \tau^{(x+1)/2})\sigma & \text{if } x \in 2\mathbb{Z}_2 + 1 \end{cases} = \sigma,$$

and then by induction

$$(\mathbb{1}^n * \sigma)^{\tau^x} = \begin{cases} ((\mathbb{1}^{n-1} * \sigma)^{\tau^{x/2}}, 1) & \text{if } x \in 2\mathbb{Z}_2 \\ (1, (\mathbb{1}^{n-1} * \sigma)^{\tau^{(x-1)/2}}) & \text{if } x \in 2\mathbb{Z}_2 + 1 \end{cases} \in \sigma^F.$$

Since T clearly centralizes τ, it follows that T normalizes G.

Next, we compute $\mathsf{Norm}(G) \cap A = TD$; indeed $\sigma^{u_x} = (\tau^{(x-1)/2}, (\tau^{(x-1)/2})^{u_x})\sigma$ belongs to G, and therefore $(x-1)/2 \in \mathbb{Z}$, so $x \in 2\mathbb{Z} + 1$ is invertible, and $x = \pm 1$.

Finally, consider an arbitrary $a \in \mathsf{Norm}(G)$. Since $\tau^a \in G$, we have $\tau^a = (\tau^{i_{1\ldots 1}}, \ldots, \tau^{i_{2\ldots 2}})\pi$ for some permutation π of the first n levels of $\mathcal{T}$. Furthermore π is 2^n-cycle, and therefore $(\tau^{2^n})^a = (\tau^j, \ldots, \tau^j)$ for some odd $j \in \mathbb{Z}$.

On the other hand, since G is layered, we may decompose $a = (b, \ldots, b)(1, g_{1\ldots 12}, \ldots, g_{2\ldots 2})\pi$ for some $g_v \in G$ and $b \in \mathsf{Norm}(G)$. Then $\tau^b = \tau^j$, and therefore $b \in TD$, so $a \in FTD$.

The last assertion follows from Theorem 4.7. Indeed in that theorem's notation $U_2 = \Omega(\mathsf{Norm}(G), G) = G$, since $[\tau^x d, \tau^y] = \tau^{2y} \notin G$ as soon as $y \notin \mathbb{Z}$, whence $U_2 \leq FT$; and $(\tau^x)^2 \notin G$ for $x \notin \mathbb{Z}$, whence $U_2 = F\langle\tau\rangle$. Therefore $\mathsf{Norm}^2(G) = \mathsf{Norm}(G)(\times U_2) = \mathsf{Norm}(G)$. $\square$

5. THE THREE GROUPS $\ddot{\Gamma}, \dot{\Gamma}, \Gamma$ ACTING ON THE TERNARY TREE

We consider in this section groups acting on the ternary tree $\mathcal{T} = X^*$ with $X = \{\mathbb{1}, \mathbb{2}, \mathbb{3}\}$, that are generated by two elements x and γ. The 3-cycle $x = (\mathbb{1}, \mathbb{2}, \mathbb{3})$ induces a rigid permutation of the subtrees $\mathbb{1}\mathcal{T}, \mathbb{2}\mathcal{T}, \mathbb{3}\mathcal{T}$ of $\mathcal{T}$ as in (1). Choose now $i, j \in \{0, 1, 2\}$, and consider the element $\gamma = (\gamma, x^i, x^j)$ acting recursively as γ, x^i, x^j on the respective subtrees $\mathbb{1}\mathcal{T}, \mathbb{2}\mathcal{T}, \mathbb{3}\mathcal{T}$. Up to conjugation by a tree isometry, it is sufficient to consider such groups with $i = 1$, and there are therefore at most three non-isomorphic such groups. These groups are known to be non-isomorphic [2]. We concentrate here on the first example, the group $\ddot{\Gamma}$ for which $\gamma = (\gamma, x, x^{-1})$. It was first studied in [16].

Recall that a group is *just-infinite* if it is infinite, but all its proper quotients are finite. The main properties of $\ddot{\Gamma}$ are:

Theorem 5.1. $\ddot{\Gamma}$ *is a just-infinite* 3-*group, and it is regular branch: we have* $\ddot{\Gamma}' > \times\ddot{\Gamma}'$ *with* $\ddot{\Gamma}/\ddot{\Gamma}' \cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$ *and* $\ddot{\Gamma}'/(\ddot{\Gamma}' \times \ddot{\Gamma}' \times \ddot{\Gamma}') \cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$.

5.1. **The original strategy.** This section describes the automorphism tower of the group $\ddot{\Gamma}$, its normalizer in the pro-3-Sylow of W, and sketches similar results for two related groups acting on the ternary tree. We start by reviewing the strategy followed by the second author in [32] to compute the automorphism group of $\ddot{\Gamma}$:

(1) Define the ascending series of subgroups $\ddot{\Gamma}^0 = \ddot{\Gamma}$ and $\ddot{\Gamma}^{n+1} = \ddot{\Gamma}^n \wr \langle x \rangle$; all these groups act level-transitively on $\mathcal{T}$. Set $\ddot{\Gamma}_* = \bigcup_{n \in \mathbb{N}} \ddot{\Gamma}^n$.

(2) Observe that $\mathsf{Rist}_{\ddot{\Gamma}}(\mathbb{1}) \wr \langle x \rangle \leq \ddot{\Gamma} \leq \ddot{\Gamma} \wr \langle x \rangle = \ddot{\Gamma}^1$. Since $\ddot{\Gamma}^1/(\mathsf{Rist}_{\ddot{\Gamma}}(\mathbb{1}) \wr \langle x \rangle)$ is a finite 3-group, it follows that $\mathsf{Norm}_{\ddot{\Gamma}^1}(\ddot{\Gamma}) \geq \ddot{\Gamma}$; check that $\mathsf{Norm}_{\ddot{\Gamma}^1}(\ddot{\Gamma}) = \ddot{\Gamma}\langle (x,x,x) \rangle$. This procedure is repeated for the pair $(\mathsf{Rist}_{\ddot{\Gamma}} \wr^n \langle x \rangle, \ddot{\Gamma}^n)$ for all $n \in \mathbb{N}$, and yields $\mathsf{Norm}_{\ddot{\Gamma}_*}(\ddot{\Gamma})$.

(3) Determine $3'$-automorphisms of $\ddot{\Gamma}$: produce a Klein 4-group acting simply on the generating set $\{x, x^{-1}, \gamma, \gamma^{-1}\}$ of $\ddot{\Gamma}$.

(4) The problem is reduced to the description of the IA-automorphisms[4] of $\ddot{\Gamma}$. Express the IA-automorphisms as (potentially infinite) words in $\mathsf{Norm}_{\ddot{\Gamma}_*}(\ddot{\Gamma})$.

(5) Eliminate the possibility of infinite words (i.e. elements of $\widehat{\mathsf{Norm}_{\ddot{\Gamma}_*}(\ddot{\Gamma})}$) by using induction on the depth function of $\ddot{\Gamma}$, defined on g as the minimal level n at which all states $g@v$ belong to $\{1, x^{\pm 1}, \gamma^{\pm 1}\}$ for all $v \in X^n$.

Theorem 5.2 ([32]). *Let t be the transposition $(2, 3)$. Set $U = \langle \triangle^n(x) : n \geq 1 \rangle$, a countably infinite elementary abelian 3-group, and $V = \{1, \tau_1, \tau_2, \tau_3\}$, a Klein group, where*

$$\tau_1 = \triangle(\tau_1)t, \quad \tau_2 = \triangle(\tau_3), \quad \tau_3 = \triangle(\tau_2)t.$$

Then we have a split extension

$$\mathsf{Aut}(\ddot{\Gamma}) = \ddot{\Gamma} : U : V.$$

Furthermore V normalizes U, and acts on it via

$$\triangle^n(x)^{\tau_1} = \triangle^n(x)^{-1}, \qquad \triangle^n(x)^{\tau_2} = \triangle^n(x)^{(-1)^n}, \qquad \triangle^n(x)^{\tau_3} = \triangle^n(x)^{-(-1)^n},$$
$$\gamma^{\tau_1} = \gamma, \qquad\qquad\qquad \gamma^{\tau_2} = \gamma^{-1}, \qquad\qquad\qquad \gamma^{\tau_3} = \gamma^{-1}.$$

Theorem 5.3. *Set $T = \langle \nabla\triangle^{2n+1}(x) : n \geq 0 \rangle$, a countably infinite elementary abelian 3-group. Then*

$$\mathsf{Aut}^2(\ddot{\Gamma}) = \mathsf{Aut}(\ddot{\Gamma}) : T.$$

Furthermore T centralizes U and V.

$$\mathsf{Aut}^3(\ddot{\Gamma}) = \mathsf{Aut}^2(\ddot{\Gamma}).$$

Remark 5.4. It follows from the proof that $\ddot{\Gamma}'$ is absolutely characteristic in $\ddot{\Gamma}$, and $\mathsf{Aut}^2(\ddot{\Gamma})$ is a subgroup of the commensurator of $\ddot{\Gamma}$.

To simplify the notations, we write $A = \mathsf{Aut}(\ddot{\Gamma})$ and $B = \mathsf{Aut}^2(\ddot{\Gamma})$. For $v \in X^n$, we also write

$$\mathsf{Stab}_{\ddot{\Gamma}}^v(n) = \{g \in \mathsf{Stab}_{\ddot{\Gamma}}(n) \mid g@v = 1\}.$$

Lemma 5.5 ([32]). *$\ddot{\Gamma}'$ is a subdirect product of $\ddot{\Gamma} \times \ddot{\Gamma} \times \ddot{\Gamma}$, and factors as*

$$\ddot{\Gamma}' = \left(\ddot{\Gamma}' \times \ddot{\Gamma}' \times \ddot{\Gamma}' \right) \langle [\gamma, x], \triangle\gamma \rangle.$$

Furthermore, $\mathsf{Cent}_W(\ddot{\Gamma}')$ is trivial.

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[4]i.e. those automorphisms which induce the identity on $\ddot{\Gamma}/\ddot{\Gamma}'$

Lemma 5.6. $A' = \ddot{\Gamma} : U$, and $A'' = \ddot{\Gamma}'$. Therefore $\ddot{\Gamma}'$ is characteristic in A.

Proof. The first assertion follows immediately from the fact that the every element of the generating set $\{\gamma, \Delta^n(x) : n \geq 0\}$ of $\ddot{\Gamma}U$ is inverted by τ_1 or τ_2. The second assertion follows from $A'' = \ddot{\Gamma}'[\ddot{\Gamma}, U]$, and $[x, \Delta^n(x)] = 1$, $[\gamma, \Delta^n(x)] = ([\gamma, \Delta^{n-1}(x)], 1, 1) \in \ddot{\Gamma}'$. $\qquad\square$

Lemma 5.7. $B \leq (A \times A \times A)\langle x, \tau_1 \rangle$.

Proof. First, note that A is saturated (the subgroups $\langle v * \ddot{\Gamma}' : |v| = n \rangle$ may serve as H_n) and weakly branch (since it contains $\ddot{\Gamma}$), and apply Theorem 3.7 to obtain $B \cong \mathsf{Norm}_W(A)$; these two groups will be identified. As $\ddot{\Gamma}'$ is B-invariant and $\mathsf{Cent}_W(\ddot{\Gamma}') = \{e\}$, we conclude that $B \leq \mathsf{Aut}(\ddot{\Gamma}')$.

Given $\phi \in B$, it can be written as $\phi = (\phi_{\mathbb{1}}, \phi_2, \phi_3)\tau_1^i x^j$ for some $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$. We conclude $(\phi_{\mathbb{1}}, \phi_2, \phi_3) \in B$. Now since $\ddot{\Gamma}'$ is a subdirect product of $\ddot{\Gamma} \times \ddot{\Gamma} \times \ddot{\Gamma}$, we obtain $\phi_x \in \mathsf{Norm}_W(\ddot{\Gamma})$ for $x \in X$. The conclusion follows. $\qquad\square$

Lemma 5.8 ([32]). *Take* $g \in \ddot{\Gamma}$. *Then* $g \in \mathsf{Stab}_{\ddot{\Gamma}}(1)$ *if and only if*

$$g = \left(c_{\mathbb{1}} \gamma^{n_1} x^{n_3 - n_2}, c_2 \gamma^{n_2} x^{n_1 - n_3}, c_3 \gamma^{n_3} x^{n_2 - n_1} \right)$$

for some $c_x \in \ddot{\Gamma}'$ *and* $n_1, n_2, n_3 \in \{0, 1, 2\}$. *We have*

$$\mathsf{Stab}_{\ddot{\Gamma}}(1) = \left(\ddot{\Gamma}' \times \ddot{\Gamma}' \times \ddot{\Gamma}' \right) \langle \Delta \gamma, (1, \gamma x^{-1}, \gamma x), (x, x^{-1}, \gamma) \rangle,$$

$$\text{and } \mathsf{Stab}_{\ddot{\Gamma}}^{\mathbb{1}}(1) = \left(1 \times \ddot{\Gamma}' \times \ddot{\Gamma}' \right) \langle (1, \gamma x^{-1}, \gamma x).$$

Lemma 5.9.

$$\mathsf{Stab}_A(1) = \mathsf{Stab}_{\ddot{\Gamma}}(1) U \langle \tau_2 \rangle,$$

$$\text{and } \mathsf{Stab}_A^{\mathbb{1}}(1) = \mathsf{Stab}_{\ddot{\Gamma}}^{\mathbb{1}}(1) \langle (1, \gamma^{-1}, \gamma) \rangle,$$

which is a subdirect product of $1 \times \ddot{\Gamma} \times \ddot{\Gamma}$.

Proof. The first statement is clear. Then take $\phi \in \mathsf{Stab}_A^{\mathbb{1}}(1)$; we can write

$$\phi = \left(c_{\mathbb{1}} \gamma^{n_1} x^{n_3 - n_2} uv, c_2 \gamma^{n_2} x^{n_1 - n_3} uv, c_3 \gamma^{n_3} x^{n_2 - n_1} uv \right)$$

for some $u \in U@\mathbb{1} = U\langle x \rangle$ and $v \in V$. Since $c_{\mathbb{1}} \gamma^{n_1} x^{n_3 - n_2} uv = 1$, we have $n_1 = 0, c_{\mathbb{1}} = 1, v = 1$, and $u \in \langle x \rangle$ with $x^{n_3 - n_2} u = 1$. Modulo $\mathsf{Stab}_{\ddot{\Gamma}}^{\mathbb{1}}(1)$, it suffices to consider $u = x$ and $n_2 = 2, n_3 = 1$, leading to $(1, \gamma^{-1}, \gamma)$.

The last assertion follows, since the second and third coordinates of $\langle (1, \gamma x^{-1}, \gamma x), (1, \gamma^{-1}, \gamma) \rangle$ each generate $\ddot{\Gamma}$. $\qquad\square$

Lemma 5.10. *Let* H *be a group such that* $A \leq H \leq (A \times A \times A)\langle x \rangle V$, *and take* $\phi \in H$. *Then* ϕ *can be reduced modulo* A *to* $(v_{\mathbb{1}}, u_2 v_2, x^i \gamma^j u_3 v_3)$.

Proof. Direct. $\qquad\square$

Lemma 5.11. *Consider* $\phi = \nabla \Delta^{2n+1}(x)$, *for* $n \geq 0$. *Then* $\phi \in B$ *and* $[UV, \phi] = 1$.

Proof. To show that $\phi \in B$ it suffices to show $[g, \phi] \in A$ for all generators g of A. First, $[x, \phi] = \Delta^{2n+2}(x)$ and $[\gamma, \phi] = [\Delta^m(x), \phi] = 1$. Then we have

$$[\tau_1, \phi] = \nabla \Delta^{2n+1}(x)^{-\tau_1} \phi = \nabla (\Delta^{2n+1}(x)^{\tau_1})\phi = \nabla \Delta^{2n+1}(x)^{-1}\phi = 1,$$

$$\text{and } [\tau_2, \phi] = \nabla \Delta^{2n+1}(x)^{-\tau_2} \phi = \nabla (\Delta^{2n+1}(x)^{-\tau_3})\phi = \nabla \Delta^{2n+1}(x)^{-1}\phi = 1,$$

since τ_1 inverts $\Delta^{2n+1}(x)$ and τ_2 fixes $\Delta^{2n+1}(x)$. $\qquad\square$

Proof of Theorem 5.3, first part. Take ϕ normalizing A, and write it, using Lemma 5.10, as $\phi = (v_\mathbb{1}, u_2 v_2, x^i \gamma^j u_3 v_3)$. We compute the commutator of ϕ with generators of A in turn:

interaction with τ_2:

$$\phi^{\tau_2} = (v_\mathbb{1}, u_2 v_2, x^i \gamma^j u_3 v_3)^{\triangle \tau_3} = (v_\mathbb{1}, u_2^{\tau_3} v_2, x^{-i} \gamma^{-j} u_3^{\tau_3} v_3),$$

$$[\tau_2, \phi^{-1}] = (1, [\tau_3, u_2^{-1}], x^{-i} \gamma^{-j} [\tau_3, u_3^{-1}] \gamma^{-j} x^{-i}) \in \mathsf{Stab}_A^\mathbb{1}(1).$$

Therefore $[\tau_3, u_2^{-1}] = [\tau_3, u_3^{-1}] = e$ and $i = j = 0$; hence $\phi = (v_\mathbb{1}, u_2 v_2, u_3 v_3)$, where $u_2, u_3 \in \mathsf{Cent}(\tau_3)$; that is, are products of $\triangle^{2n+1}(x)$. Now using T we may reduce ϕ further to the form $\phi = (v_\mathbb{1}, v_2, uv_3)$, for some $u \in U$ a product of $\triangle^{2n+1}(x)$.

interaction with τ_1:

$$\phi^{\tau_1} = (v_\mathbb{1}, v_2, uv_3)^{\triangle(\tau_1)t} = (v_\mathbb{1}, uv_3, v_2)^{\triangle \tau_1} = (v_\mathbb{1}, u^{-1} v_3, v_2),$$

$$[\tau_1, \phi^{-1}] = (v_\mathbb{1}, u^{-1} v_3, v_2)(v_\mathbb{1}, v_2, (uv_3)^{-1}) = (1, u^{-1} v_3 v_2, v_2 v_3 u^{-1}) \in \mathsf{Stab}_A^\mathbb{1}(1).$$

Therefore $u = 1$, $v_2 = v_3$, and ϕ is reduced to the form $\phi = (v_\mathbb{1}, v_2, v_2)$.

interaction with x:

$$[x, \phi^{-1}]^x = (1, v_2 v_\mathbb{1}, v_\mathbb{1} v_2) \in \mathsf{Stab}_A^\mathbb{1}(1).$$

Therefore $v_\mathbb{1} = v_2$, and ϕ is reduced to $\phi = (v, v, v)$. If $v = \tau_3$ we have $\phi = \tau_2 \in A$, so it suffices to consider $\phi = \triangle \tau_1$.

interaction with γ:

$$[\triangle \tau_1, \gamma^{-1}] = (\gamma, x^{-1}, x)(\gamma, x, x^{-1})^{-1} = (1, x, x),$$

but this last element is not in $\mathsf{Stab}_A^\mathbb{1}(1)$; therefore, this last case is impossible, and these successive steps have reduced ϕ to 1.

$\square$

Lemma 5.12. $B' = \ddot{\Gamma} : U = A'$, *and* $B'' = \ddot{\Gamma}' = A''$.
$\mathsf{Stab}_B^\mathbb{1}(1) = \mathsf{Stab}_A^\mathbb{1}(1) : T$.
B *is a saturated subgroup of* W, *and* $\mathsf{Aut}\, B \leq (A \times A \times A)\langle x, \tau_1 \rangle$.

Proof. Direct. $\square$

Proof of Theorem 5.3, second part. Let ϕ normalize B. As before, we may assume $\phi = (v_\mathbb{1}, u_2 v_2, x^i \gamma^j u_3 v_3)$. Now,

$$[\tau_2, \phi^{-1}] = (1, [\tau_3, u_2^{-1}], x^{-i} \gamma^{-j} [\tau_3, u_3^{-1}] \gamma^{-j} x^{-i}) \in \mathsf{Stab}_B^\mathbb{1}(1).$$

Therefore $i = j = 0$, and $(1, [\tau_3, u_2^{-1}], [\tau_3, u_3^{-1}]) \in T$. On one hand, T is centralized by τ_3; but on the other hand, $[\tau_3, u_2^{-1}]$ and $[\tau_3, u_3^{-1}]$ are inverted by τ_3, since τ_3 is an involution. Since T has no 2-torsion, it follows that $[\tau_3, u_2^{-1}] = [\tau_3, u_3^{-1}] = 1$. We then have $\phi = (v_\mathbb{1}, u_2 v_2, u_3 v_3)$ where u_2, u_3 are centralized by τ_3. Using $(1, u_2, u_2^{-1}) \in T$, we may reduce to the case $\phi = (v_\mathbb{1}, v_2, uv_3)$.

We next compute

$$\phi^{\tau_1} = (v_\mathbb{1}, (uv_3)^{\tau_1}, v_2) = (v_\mathbb{1}, u^{-1} v_3, v_2),$$

$$[\tau_1, \phi^{-1}] = (1, u^{-1} v_3 v_2, v_2 v_3 u^{-1}) \in \mathsf{Stab}_B^\mathbb{1}(1).$$

Therefore $v_2 = v_3$ and $[\tau_1, \phi^{-1}] = (1, u^{-1}, u^{-1}) \in UT$, hence $u = 1$ and the proof is finished. $\square$

5.2. **The normalizer tower.** The reason the automorphism tower of $\ddot{\Gamma}$ stops after 2 steps can be attributed to the involutions in V; just in the same way as the lower central series of $\mathsf{Aut}(\ddot{\Gamma})$ stabilizes at $\ddot{\Gamma}:U$ and that of $\mathsf{Aut}^2(\ddot{\Gamma})$ stabilizes at $\ddot{\Gamma}:UT$, the normalizer tower may not be extended beyond V.

However, since $\ddot{\Gamma}$ is a subgroup of the pro-3-Sylow $C = \widehat{\langle x \rangle}_*$ of W, we may consider its normalizer tower within C. For this purpose, consider the set $\Omega = \left\{ Z\,x \,\middle|\, Z \in \{\triangle, \nabla, \dashv\}^* \right\}$, with the notation of Definition 3.2. The set Ω will serve as a basis for the quotients of successive terms of the normalizer tower of $\ddot{\Gamma}$. Define a rank function on Ω by

$$\operatorname{rank} x = 1, \qquad\qquad \operatorname{rank}(\dashv x) = 3\operatorname{rank} x,$$
$$\operatorname{rank}(\nabla x) = 3\operatorname{rank} x - 1, \qquad\qquad \operatorname{rank}(\triangle x) = 3\operatorname{rank} x - 2,$$

and define $A_n = \langle x, \gamma, Z\,x \text{ such that } \operatorname{rank}(Z\,x) \leq n \rangle$. The elements of Ω of low rank are as given below:

rank		elements					
1	x	$\triangle x$	$\triangle\triangle x$	$\triangle\triangle\triangle x$	$\triangle\triangle\triangle\triangle x$	$\cdots$	$\triangle^n x$
2		∇x	$\nabla\triangle x$	$\nabla\triangle\triangle x$	$\nabla\triangle\triangle\triangle x$	$\cdots$	$\nabla\triangle^n x$
3		$\dashv x$	$\dashv\triangle x$	$\dashv\triangle\triangle x$	$\dashv\triangle\triangle\triangle x$	$\cdots$	$\dashv\triangle^n x$
4			$\triangle\nabla x$	$\triangle\nabla\triangle x$	$\triangle\nabla\triangle\triangle x$	$\cdots$	$\triangle\nabla\triangle^n x$
5			$\nabla\nabla x$	$\nabla\nabla\triangle x$	$\nabla\nabla\triangle\triangle x$	$\cdots$	$\nabla\nabla\triangle^n x$
6			$\dashv\nabla x$	$\dashv\nabla\triangle x$	$\dashv\nabla\triangle\triangle x$	$\cdots$	$\dashv\nabla\triangle^n x$
7			$\triangle\dashv x$	$\triangle\dashv\triangle x$	$\triangle\dashv\triangle\triangle x$	$\cdots$	$\triangle\dashv\triangle^n x$
8			$\nabla\dashv x$	$\nabla\dashv\triangle x$	$\nabla\dashv\triangle\triangle x$	$\cdots$	$\nabla\dashv\triangle^n x$
9			$\dashv\dashv x$	$\dashv\dashv\triangle x$	$\dashv\dashv\triangle\triangle x$	$\cdots$	$\dashv\dashv\triangle^n x$
10				$\triangle\triangle\nabla x$	$\triangle\triangle\nabla\triangle x$	$\cdots$	$\triangle\triangle\nabla\triangle^n x$

The next rows follow the same pattern: elements of rank n are determined by writing backwards $n-1$ in base 3, using the symbols $\triangle = 0$, $\nabla = 1$ and $\dashv = 2$.

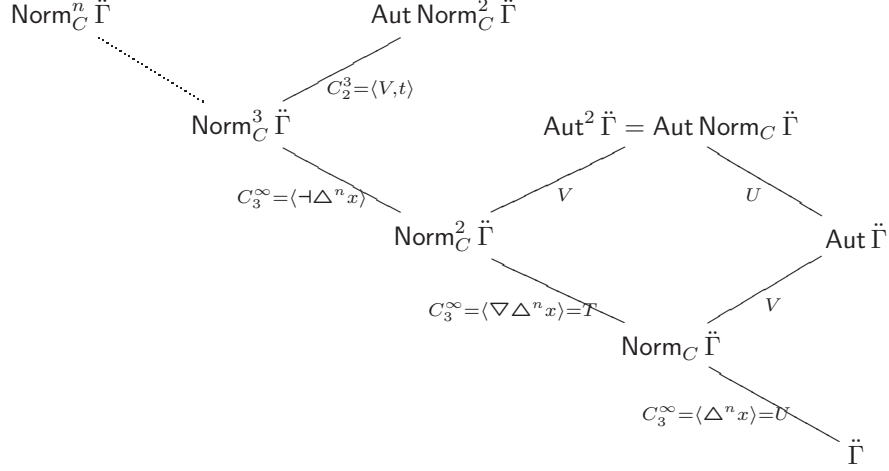**Theorem 5.13.** *The following facts hold for all $n \geq 1$:*

(1) $\mathsf{Norm}_C^n(\ddot{\Gamma}) \geq A_n$;

(2) *each $\mathsf{Norm}_C^{n+1}(\ddot{\Gamma})/\mathsf{Norm}_C^n(\ddot{\Gamma})$ is an infinite countable elementary abelian 3-group;*

(3) $\mathsf{Norm}_C^\alpha(\ddot{\Gamma}) = \langle \gamma, \Omega \rangle = \ddot{\Gamma}_*$ *for $\alpha \geq \omega$.*

Proof. Extend the rank function to $\Omega \cup \{\gamma\}$, by $\operatorname{rank} \gamma = 0$. Inductive computations then show that $\operatorname{rank}[u, v] \leq \max\{\operatorname{rank} u, \operatorname{rank} v\} - 1$ for all $u \in \Omega \cup \{\gamma\}$ and $v \in \Omega$ of rank at least 1. For example, for any $Y, Z \in \{\triangle, \nabla, \dashv\}^*$:

$$\operatorname{rank}[\nabla Y x, \triangle Z x] = \operatorname{rank} \nabla[Y x, Z x]$$
$$\leq 3\operatorname{rank}[Y x, Z x] - 1$$
$$\leq 3(\max\{\operatorname{rank}(Y x), \operatorname{rank}(Z x)\} - 1) - 1$$
$$\leq \operatorname{rank}\max\{\operatorname{rank}(\nabla Y x), \operatorname{rank}(\nabla Y x)\} - 1,$$
$$\text{and } \operatorname{rank}[x, \nabla Z x] = \operatorname{rank}\triangle Z x = \operatorname{rank}(\nabla Z x) - 1.$$

We therefore have $A_n \lhd A_{n+1}$ with abelian quotients, so the tower A_n grows more slowly than the normalizer tower.

Finally, the tower of $\ddot{\Gamma}$ stops at level ω by Theorem 3.15. $\square$

$$\mathsf{Norm}_C^n\,\ddot{\Gamma} \qquad\qquad \mathsf{Aut}\,\mathsf{Norm}_C^2\,\ddot{\Gamma}$$

(diagram of lattice)

$C_2^3 = \langle V, t\rangle$

$$\mathsf{Norm}_C^3\,\ddot{\Gamma} \qquad\qquad \mathsf{Aut}^2\,\ddot{\Gamma} = \mathsf{Aut}\,\mathsf{Norm}_C\,\ddot{\Gamma}$$

$C_3^\infty = \langle\dashv\triangle^n x\rangle$ $\qquad\qquad V \qquad\qquad U$

$$\mathsf{Norm}_C^2\,\ddot{\Gamma} \qquad\qquad\qquad\qquad \mathsf{Aut}\,\ddot{\Gamma}$$

$C_3^\infty = \langle\nabla\triangle^n x\rangle = \mathcal{T}$ $\qquad\qquad V$

$$\mathsf{Norm}_C\,\ddot{\Gamma}$$

$C_3^\infty = \langle\triangle^n x\rangle = \mathcal{U}$

$$\ddot{\Gamma}$$

5.3. The Fabrykowski-Gupta group.

This group was introduced in [10] as an alternate construction of a group of intermediate growth. Like $\ddot{\Gamma}$, it acts on the ternary tree $\{\mathbb{1}, \mathbb{2}, \mathbb{3}\}^*$. Let $x = (\mathbb{1}, \mathbb{2}, \mathbb{3})$ be the 3-cycle acting on $\mathcal{T}$ according to (1), and define recursively $\gamma = (\gamma, x, 1)$. The Fabrykowski-Gupta group is $\Gamma = \langle x, \gamma\rangle$. It is a regular branch group; we have $\Gamma/\Gamma' \cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$, and $\Gamma'/(\Gamma' \times \Gamma' \times \Gamma') \cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$.

The generators x, γ of Γ have order 3, and $\langle x\gamma, \gamma x\rangle$ is a torsion-free normal subgroup of Γ of index 3.

Theorem 5.14. *Set $U = \langle\triangle^n(x) : n \geq 1\rangle$, a countably infinite elementary abelian 3-group. Then we have a split extension*

$$\mathsf{Aut}(\Gamma) = \Gamma : U.$$

Proof. The proof follows the same scheme as that of Theorem 5.3 and is omitted. $\qquad\square$

5.4. The group $\dot{\Gamma}$.

This group also acts on the ternary tree, and is generated by $x = (\mathbb{1}, \mathbb{2}, \mathbb{3})$ and $\gamma = (\gamma, x, x)$. The group $\dot{\Gamma}$ was first studied in [1,2], where some of its elementary properties were proven.

Note that the group $\dot{\Gamma}$ is not branch, but is a regular weakly branch group. Consider the subgroup $K = \langle x^{-1}\gamma, \gamma x^{-1}\rangle$; then K is a torsion-free normal subgroup of index 3. We have $\dot{\Gamma}/\dot{\Gamma}' \cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$, and $\dot{\Gamma}'/K' \cong \mathbb{Z}^2$, and $K'/(K' \times K' \times K') \cong \mathbb{Z}^2$.

Theorem 5.15. *Let t be the transposition $(\mathbb{2}, \mathbb{3})$ acting on $\mathcal{T}$ according to (1). Let $U = \langle\triangle^n(x\gamma^{-1}) : n \geq 1\rangle$ be a countably infinite elementary abelian 3-group, and let $V = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein group defined by*

$$\tau_1 = \triangle(\tau_1)t, \quad \tau_2 = \triangle(\tau_1), \quad \tau_3 = t.$$

Then we have a non-split extension

$$\mathsf{Aut}(\dot{\Gamma}) = (\dot{\Gamma} \cdot U) : V,$$

where V acts on U, for $n > 0$, by

$$x^{\tau_1} = x^{-1}, \qquad\qquad x^{\tau_2} = x, \qquad\qquad x^{\tau_3} = x^{-1},$$
$$\triangle^n(x)^{\tau_1} = \triangle^n(x)^{-1}, \qquad \triangle^n(x)^{\tau_2} = \triangle^n(x)^{-1}, \qquad \triangle^n(x)^{\tau_3} = \triangle^n(x),$$
$$\gamma^{\tau_1} = \gamma^{-1}, \qquad\qquad \gamma^{\tau_2} = \gamma^{-1}, \qquad\qquad \gamma^{\tau_3} = \gamma.$$

Let $T = \langle \nabla \triangle^{2n+1}(x) : n \geq 0 \rangle$ be a countably infinite elementary abelian 3*-group. Then*

$$\mathsf{Aut}^3(\dot{\Gamma}) = \mathsf{Aut}^2(\dot{\Gamma}) = \mathsf{Aut}(\dot{\Gamma}) : T.$$

Furthermore T centralizes U and V.

Proof. The proof follows the same scheme as that of Theorem 5.3 and is omitted. $\qquad\square$

6. The First Grigorchuk Group

This section describes automorphism tower of the first Grigorchuk group. Consider the regular binary tree X^* with $X = \{\mathbb{1}, \mathbb{2}\}$, denote by σ the non-trivial permutation $(\mathbb{1}, \mathbb{2})$ of X and define isometries a, b, c, d of $\mathcal{T}$ by

$$(x_1 \ldots x_m)^a = x_1^\sigma x_2 \ldots x_m,$$

$$b = (a, c) = \prod_{i \not\equiv 0 \mod 3} 2^i \mathbb{1} * a,$$

$$c = (a, d) = \prod_{i \not\equiv 2 \mod 3} 2^i \mathbb{1} * a,$$

$$d = (1, b) = \prod_{i \not\equiv 1 \mod 3} 2^i \mathbb{1} * a.$$

The Grigorchuk group [13,14] is $\mathfrak{G} = \langle S \rangle$, with $S = \{a, b, c, d\}$. It is a regular branch group. Set $B = \mathsf{Rist}_\mathfrak{G}(\mathbb{1})@\mathbb{1} = \langle b \rangle^G$ and $K = \mathsf{Rist}_\mathfrak{G}(\mathbb{1}^2)@\mathbb{1}^2 = \langle [a, b] \rangle^\mathfrak{G}$. Then B has index 8 in $\mathfrak{G}$, and K has index 16; and $K/(K \times K) \cong \mathbb{Z}/4$.

The automorphism group of $\mathfrak{G}$ was computed in [15], where it was shown, in a manner similar to the one employed in [32], that $\mathsf{Out}(\mathfrak{G})$ is an infinite elementary 2-group, generated by $\triangle^n \dashv adad$ for all $n \geq 1$, in the notation of Definition 3.2. We will give below a complete description of the tower, without relying on the above result. In order to illustrate our method, we first examine more closely the first terms of the automorphism tower.
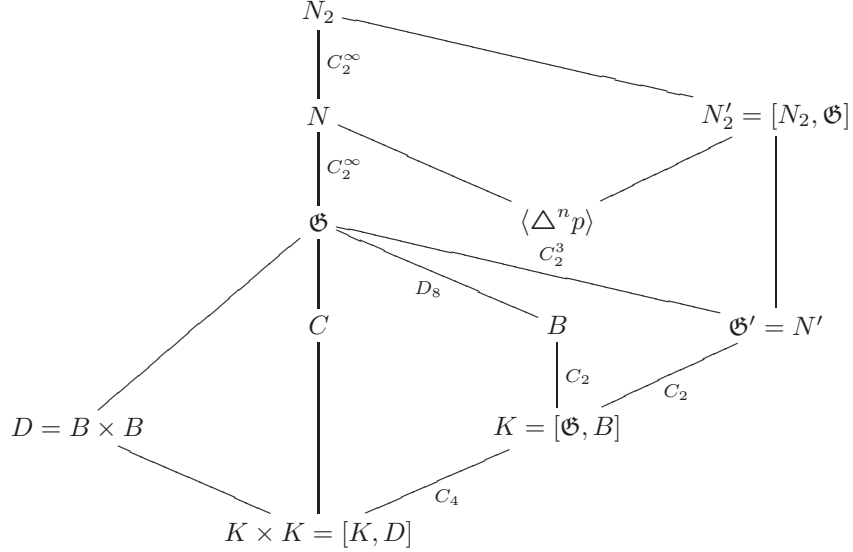
Let us define inductively the normalizer tower by $N_0 = \mathfrak{G}$ and $N_{n+1} = \mathsf{Norm}_W(N_n)$. Then $N_1 = \langle a, b, c, d, \triangle^n \dashv adad \rangle$. There are clearly some elements of W outside N_1 that normalize N_1; for instance, $\dashv ad$. In fact, a little computation shows that the elements $\dashv \triangle^n \dashv adad$ for all $n \in \mathbb{N}$ also normalize N_1: indeed for all $m, n \geq 0$:

$$[\dashv\triangle^n\dashv adad, \triangle^m\dashv adad] = \begin{cases} \dashv\triangle^{m-1}\dashv[\triangle^{n-m-1}\dashv adad, \triangle b] = 1 & \text{if } m < n+1; \\ 1 & \text{if } m = n+1, \\ \dashv\triangle^n\dashv[\triangle b, \triangle^{m-n-2}\dashv adad] = 1 & \text{if } m > n+1; \end{cases}$$

$$[\dashv\triangle^n\dashv adad, a] = \triangle^{n+1}\dashv adad \in N_1;$$

$$[\dashv\triangle^n\dashv adad, b] = [\dashv\triangle^n\dashv adad, c] = [\dashv\triangle^n adad, d] = 1.$$

Similarly, we have $\dashv ad$ in N_2.

Proceeding, we see $\dashv a$ in N_3, and therefore N_3 contains $\times\mathfrak{G}$. We may also check that N_3 contains some other elements, like $\triangle\dashv ad$ and $\triangle\dashv\dashv adad$. Based on the behaviour prescribed by Theorem 4.7 for layered groups, we may "guess" that N_3 is $(\times\mathfrak{G})(\triangle N_1)\langle a \rangle$.

Actually, the tower of automorphisms of $\mathfrak{G}$ follows a pattern that is best described by powers of 2. We describe it in the same language as the normalizer tower of $\ddot{\Gamma}$ given in Subsection 5.2.

$$N_2 \quad C_2^\infty \quad N \quad N_2' = [N_2, \mathfrak{G}]$$

Diagram (Hasse-type lattice) with nodes and edge labels:

- N_2 — C_2^∞ — N
- N — C_2^∞ — $\mathfrak{G}$
- $N_2' = [N_2, \mathfrak{G}]$
- $\langle \triangle^n p \rangle$, edge C_2^3
- $\mathfrak{G}$ — D_8 — C — B — $\mathfrak{G}' = N'$
- $D = B \times B$, $K = [\mathfrak{G}, B]$ with edges C_2, C_2
- $K \times K = [K, D]$ with edge C_4

Let us define

$$p = \dashv adad, \qquad q = \dashv ad, \qquad r = \dashv a,$$

and consider the subset $\Omega = \{Zp, Zq, Zr \mid Z \in \{\triangle, \dashv\}^*\}$ of $\mathfrak{G}_*$, which will serve as a basis for the quotients of successive terms of the automorphism tower of $\mathfrak{G}$. Define a rank on Ω by

$$\operatorname{rank} p = 1, \qquad \operatorname{rank} q = 2, \qquad \operatorname{rank} r = 3$$

and inductively

$$\operatorname{rank}(\dashv x) = 2\operatorname{rank} x, \qquad \operatorname{rank}(\triangle x) = 2\operatorname{rank} x - 1.$$

Then $\langle \Omega \cup S \rangle = \mathfrak{G}_*$ is the layered closure of $\mathfrak{G}$. Extend the rank function to $\Omega \cup S$ by $\operatorname{rank}(s) = 0$ for all $s \in S$, and define the following subgroups of $\mathfrak{G}_*$:

$$A_n = \langle x \in \Omega \cup S : \operatorname{rank} x \leq n \rangle.$$

The elements of Ω of low rank are as given below:

rank	elements							
1			p	$\triangle p$	$\triangle\triangle p$	$\triangle\triangle\triangle p$	$\triangle\triangle\triangle\triangle p$	$\cdots$ $\triangle^n p$
2		q	$\dashv p$	$\dashv\triangle p$	$\dashv\triangle\triangle p$	$\dashv\triangle\triangle\triangle p$		$\cdots$ $\dashv\triangle^n p$
3	r	$\triangle q$		$\triangle\dashv p$	$\triangle\dashv\triangle p$	$\triangle\dashv\triangle\triangle p$		$\cdots$ $\triangle\dashv\triangle^n p$
4		$\dashv q$		$\dashv\dashv p$	$\dashv\dashv\triangle p$	$\dashv\dashv\triangle\triangle p$		$\cdots$ $\dashv\dashv\triangle^n p$
5	$\triangle r$	$\triangle\triangle q$			$\triangle\triangle\dashv p$	$\triangle\triangle\dashv\triangle p$		$\cdots$ $\triangle\triangle\dashv\triangle^n p$
6	$\dashv r$	$\dashv\triangle q$			$\dashv\triangle\dashv p$	$\dashv\triangle\dashv\triangle p$		$\cdots$ $\dashv\triangle\dashv\triangle^n p$
7		$\triangle\dashv q$			$\triangle\dashv\dashv p$	$\triangle\dashv\dashv\triangle p$		$\cdots$ $\triangle\dashv\dashv\triangle^n p$
8		$\dashv\dashv q$			$\dashv\dashv\dashv p$	$\dashv\dashv\dashv\triangle p$		$\cdots$ $\dashv\dashv\dashv\triangle^n p$
9	$\triangle\triangle r$	$\triangle\triangle\triangle q$				$\triangle\triangle\triangle\dashv p$		$\cdots$ $\triangle\triangle\triangle\dashv\triangle^n p$
10	$\dashv\triangle r$	$\dashv\triangle\triangle q$				$\dashv\triangle\triangle\dashv p$		$\cdots$ $\dashv\triangle\triangle\dashv\triangle^n p$

The next rows follow the same pattern: elements of rank n are determined by writing backwards $n - 1$ in base 2, using the symbols $\triangle = 0$ and $\dashv = 1$.

Theorem 6.1. *We have for all $i \in \mathbb{N}$*

$$\operatorname{Aut}^i(\mathfrak{G}) = \langle a, b, c, d, x \in \Omega \text{ with } \operatorname{rank} x \leq i \rangle.$$

In particular, $\mathsf{Aut}^{3\cdot 2^{n-1}}(\mathfrak{G})$ *contains* $v * \mathfrak{G}$ *for all* $v \in X^n$, *since* $\mathrm{rank}(\dashv^n a) = 3 \cdot 2^{n-1}$.

For all $i \in \mathbb{N}$, *we have* $\mathsf{Out}(\mathsf{Aut}^i(\mathfrak{G})) = (\mathbb{Z}/2)^\omega$ *generated by the* $x \in \Omega$ *of rank* $i + 1$.

We have $\mathsf{Aut}^\alpha(\mathfrak{G}) = \langle \Omega \rangle = \mathfrak{G}_*$ *for all* $\alpha \geq \omega$.

Lemma 6.2. *For all* $s \in S$, *the subgroups* $\langle s \rangle^{\mathfrak{G}}$ *are all characteristic and distinct.*

Proof. The level stabilizer $\mathsf{Stab}_{\mathfrak{G}}(n)$ is characteristic in $\mathfrak{G}$ for all n, by Theorem 3.7. The subgroup $\langle s \rangle^{\mathfrak{G}}$ contains $\mathsf{Stab}_{\mathfrak{G}}(4)$, by direct computation [2, Lemma 4.1]. To check that $\langle s \rangle^{\mathfrak{G}}$ is normal in $\mathsf{Aut}(\mathfrak{G})$, it is therefore sufficient to check that $\langle s \rangle^{\mathfrak{G}} / \mathsf{Stab}_{\mathfrak{G}}(4)$ is normal in $\mathsf{Aut}(\mathfrak{G}) / \mathsf{Stab}_{\mathfrak{G}}(4)$.

Now $\mathsf{Aut}(\mathfrak{G}) < W$ by Theorem 3.7, so all computations may be performed in $\overline{W} = W / \mathsf{Stab}_W(4) = \wr^4 C_2$, a 2-Sylow subgroup of $\mathsf{Sym}(2^4)$. The image $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ has order 2^{12}, and the image of $\mathsf{Aut}(\mathfrak{G})$ lies in $\overline{N} = \mathsf{Norm}_{\overline{W}}(\overline{\mathfrak{G}})$, a group of order 2^{13}. It is then routine to check that $\langle s \rangle^{\mathfrak{G}} / \mathsf{Stab}_{\mathfrak{G}}(4)$ is normal in $\overline{N}$.

These computations were performed using the computer software GAP [12], by expressing $\overline{W}$ as a permutation group on 16 points.[5]   □

Define a norm on $\mathfrak{G}_*$ by setting

$$\|g\| = \min \left\{ n + \mathrm{rank}(\omega_1) + \cdots + \mathrm{rank}(\omega_n) \mid g = \omega_1 \ldots \omega_n, \ \omega_i \in \Omega \cup S \right\}.$$

Note that the restriction of this norm to $\mathfrak{G}$ coincides with the norm given by (2), but we shall not need this fact. We have a contraction property on $\mathfrak{G}_*$ as in $\mathfrak{G}$:

Lemma 6.3. *For all* $g \in \mathfrak{G}_*$ *and* $x \in X$ *we have* $|g@x\| \leq \frac{1}{2}(\|g\| + 1)$.

Define next a norm on the set of group homomorphisms $\{\phi : \mathfrak{G} \to \mathfrak{G}_*\}$ by

$$|||\phi||| = \sum_{s \in S} \|\phi(s)\|.$$

Lemma 6.4. $\mathsf{Aut}^\alpha(\mathfrak{G}) \cong \mathsf{Norm}_{\mathfrak{G}_*}^\alpha(\mathfrak{G})$ *for all ordinals* $\alpha \leq \omega$.

Proof. In [21, Theorem 8.5], the authors show that $\mathfrak{G}$ is saturated, with the subgroups $(\mho_2)^n(\mathfrak{G})$ fixing X^n and acting transitively on each subtree at level n. Furthermore, $\mathfrak{G}$ is weakly branch. By Lemma 3.9, the groups $\mathsf{Aut}^\alpha(\mathfrak{G})$ are also saturated and weakly branch, so Theorem 3.7 applies, and $\mathsf{Aut}^{\alpha+1}(\mathfrak{G}) \leq \mathsf{Norm}_W(\mathsf{Aut}^\alpha(\mathfrak{G}))$ for all ordinals α.

We proceed by induction on α. Take $\phi \in \mathsf{Aut}^\alpha(\mathfrak{G})$. First, we may suppose $\phi \in W$ by the above. We construct inductively elements $\phi_n \in W$ for all $n \in \mathbb{N}$ by setting $\phi_0 = \phi$ and decomposing $\phi_n = (\lambda_n, \rho_n) a^{\varepsilon_n}$. We then note that $\lambda_n, \rho_n : \mathfrak{G} \to \mathfrak{G}_*$ since $\mathsf{Stab}_{\mathfrak{G}}(1)$ is subdirect in $\mathfrak{G} \times \mathfrak{G}$. We set ϕ_{n+1} to be that among λ_n, ρ_n of smallest $|||$-norm, and proceed.

We may assume $\varepsilon_n = 0$, up to multiplying by an element on $\mathfrak{G}_*$. Suppose ρ_n is of minimal norm; then

$$
\begin{aligned}
|||\phi_{n+1}||| &= \|\phi_{n+1}(a)\| + \|\phi_{n+1}(b)\| + \|\phi_{n+1}(c)\| + \|\phi_{n+1}(d)\| \\
&= \|\phi_n(axa)@2\| + \|\phi_n(d)@2\| + \|\phi_n(b)@2\| + \|\phi_n(c)@2\| \\
&\leq \frac{1}{2}\left(2\|\phi_n a\| + \|\phi_n x\| + \|\phi_n b\| + \|\phi_n c\| + \|\phi_n d\| + 1\right) \\
&< |||\phi_n|||
\end{aligned}
$$

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[5]Other ad hoc arguments would be preferable for manual computation; for instance, list all quotients of $\mathfrak{G}$ of order 8 and 16 to check that D is characteristic, and that if B, C are not characteristic, then there is an automorphism exchanging them. This last possibility is eliminated by computing the quotients $B / \mathsf{Stab}_{\mathfrak{G}}(3) = D_8 \times C_2$ and $C / \mathsf{Stab}_{\mathfrak{G}}(3) = C_2^4$.

for any $x \in \{b, c\}$; a similar equation holds if λ_n is of minimal norm. Therefore $|||\phi_n||| = 4$ for n large enough, and by Lemma 6.2 we have $\phi_n = 1$.

Now for $m = n - 1, n - 2, \ldots, 0$ we have $[\dashv\phi_m, a] = (\lambda_m^{-1}\rho_n, \rho_m\lambda_n^{-1}) \in \mathsf{Aut}^\alpha(\mathfrak{G}) \subseteq \mathfrak{G}_*$, and since one of λ_m, ρ_m is ϕ_{m+1} and therefore belongs to $\mathfrak{G}_*$, we obtain that both of them belong to $\mathfrak{G}_*$, and therefore ϕ_m belongs to $\mathfrak{G}_*$. We have shown $\phi_0 \in \mathfrak{G}_*$. $\square$

Proof of Theorem 6.1. By Lemma 6.4, the automorphism tower of $\mathfrak{G}$ is contained in $\mathfrak{G}_*$. We prove the theorem by showing that the generators of $\mathsf{Out}(A_n)$ are precisely the elements of rank $n + 1$; for this purpose, we show that

   (1) elements of rank $n + 1$ normalize A_n, and
   (2) no element of rank $n + 2$ or greater normalizes A_n.

It then follows that $A_n = \mathsf{Norm}_{\mathfrak{G}_*}^n(\mathfrak{G})$.

The following inductive computations show that $\mathrm{rank}[x, y] \le \max\{\mathrm{rank}\, x, \mathrm{rank}\, y\} - 1$ for $x \in \Omega$ and $y \in \Omega \cup \mathfrak{G}$:

$$\mathrm{rank}[\dashv x, \dashv y] = \mathrm{rank} \dashv [x, y] = 2\,\mathrm{rank}[x, y] \le 2(\max\{\mathrm{rank}\, x, \mathrm{rank}\, y\} - 1)$$
$$\le \max\{\mathrm{rank} \dashv x, \mathrm{rank} \dashv y\} - 1;$$
$$\mathrm{rank}[\triangle x, \dashv y] = \mathrm{rank} \dashv [x, y] = 2\,\mathrm{rank}[x, y] \le 2(\max\{\mathrm{rank}\, x, \mathrm{rank}\, y\} - 1)$$
$$\le \max\{\mathrm{rank} \triangle x, \mathrm{rank} \dashv y\} - 1;$$
$$\mathrm{rank}[\triangle x, \triangle y] = \mathrm{rank} \triangle [x, y]$$
$$= 2\,\mathrm{rank}[x, y] - 1 \le 2(\max\{\mathrm{rank}\, x, \mathrm{rank}\, y\} - 1) - 1$$
$$\le \max\{\mathrm{rank} \triangle x, \mathrm{rank} \triangle y\} - 1;$$
$$\mathrm{rank}[\dashv x, a] = \mathrm{rank} \triangle x = \mathrm{rank}(\dashv x) - 1;$$
$$\mathrm{rank}[\triangle x, a] = \mathrm{rank}\, 1 = 0;$$
$$\mathrm{rank}[\dashv x, b] = \mathrm{rank} \dashv [x, a] = 2\,\mathrm{rank}[x, a] \le 2(\mathrm{rank}\, x - 1) \le \mathrm{rank}(\dashv x) - 1;$$
$$\mathrm{rank}[\triangle x, b] = \mathrm{rank}(\dashv [x, a])(1, [x, c]) \le \max\{\mathrm{rank}[x, a], \mathrm{rank}[x, c]\}$$
$$\le \mathrm{rank}(\triangle x) - 1;$$
$$[p, a] = (ac)^8 \in \mathfrak{G}, \text{ and more generally } [p, s] \in \mathfrak{G} \text{ for } s \in S;$$
$$\mathrm{rank}[\dashv x, r] = \mathrm{rank} \dashv [x, adad] \le \mathrm{rank}(\dashv x) - 1;$$
$$[q, s] \in \mathfrak{G}\langle p \rangle \text{ for } s \in S;$$
$$[r, s] \in \mathfrak{G}\langle q \rangle \text{ for } s \in S;$$
$$[p, q] = 1, \quad [p, r] = 1, \quad [q, r] = p.$$

Finally, the tower of $\mathfrak{G}$ stops at level ω by Theorem 3.15. $\square$

We note by direct computation that the "branching subgroup" K satisfies the property that $\times^i(K)$ is characteristic in $\mathsf{Aut}^{2^i}(\mathfrak{G})$. It then follows that $\mathfrak{G} \cap \mathfrak{G}^\alpha$ has finite index in $\mathfrak{G}$ for all $\alpha \in \mathsf{Aut}^i(G)$.

More generally, we have shown for $G \in \{\mathfrak{G}, \ddot{\Gamma}\}$ that $\mathsf{Aut}^{i+1}(G)/\mathsf{Aut}^i(G)$ is solvable (actually, of degree ≤ 2) for all $i \in \mathbb{N}$. Therefore, there exists $k \in \mathbb{N}$ such that $(\mathsf{Aut}^i(G))^{(k)} \le G$, where $H^{(k)}$ denotes the kth term in the derived series of H. Since G is just-infinite and not solvable, it follows that $G \cap G^\alpha$ has finite index in both G and G^α, for all $\alpha \in \mathsf{Aut}^\omega(G)$. Hence, $\mathsf{Aut}^\omega(G)$ is a subgroup of the commensurator of G in W.

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